Math 54: Quiz #4 Solutions

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Please give neat and organized answers. Whenever applicable (especially for computational questions), make it clear what strategy you are using.

Problem 1

Let
$$\mathbf{b}_1 = \begin{bmatrix} 7 \\ -2 \end{bmatrix},\ \mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ \mathbf{c}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix},\ \mathbf{c}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$
Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$. Check that $\mathcal{B}$ and $\mathcal{C}$ are bases for $\mathbb{R}^2$. Then compute the change-of-coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$.

Solution: Note that $\mathcal{B}$ will be as basis for $\mathbb{R}^2$ as long as $\mathbf{b}_1$ and $\mathbf{b}_2$ are linearly independent. This can be checked by showing that all of the columns of the matrix $[\mathbf{b}_1\ \mathbf{b}_2]$ are pivot columns, but we can also just observe that neither $\mathbf{b}_1$ nor $\mathbf{b}_2$ is a scalar multiple of the other, which is sufficient to show the linear independence of two vectors. Similarly, neither $\mathbf{c}_1$ nor $\mathbf{c}_2$ is a scalar multiple of the other, so $\mathcal{C}$ is also a basis for $\mathbb{R}^2$.

Next we construct the matrix $[\mathbf{c}_1\ \mathbf{c}_2 \,|\, \mathbf{b}_1\ \mathbf{b}_2]$ and row-reduce to reveal $[I_2 \,|\, P_{\mathcal{C}\leftarrow\mathcal{B}}]$:
$$\begin{aligned}
\begin{bmatrix} 4 & 5 & 7 & 2 \\ 1 & 2 & -2 & -1 \end{bmatrix} &\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & -2 & -1 \\ 4 & 5 & 7 & 2 \end{bmatrix} \\
&\xrightarrow{R_2 \leftarrow R_2 - 4R_1} \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & -3 & 15 & 6 \end{bmatrix} \\
&\xrightarrow{R_2 \leftarrow -\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & -5 & -2 \end{bmatrix} \\
&\xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 8 & 3 \\ 0 & 1 & -5 & -2 \end{bmatrix},
\end{aligned}$$
so $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$.

Problem 2

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation, and suppose that T is onto. What is $\dim(\ker(T))$?

Solution: By the 'rank-nullity' theorem, $\dim(\ker(T)) + \dim(\text{Range}(T)) = 4$. Since T is onto, $\text{Range}(T) = \mathbb{R}^3$, so $\dim(\text{Range}(T)) = 3$. Thus $\dim(\ker(T)) = 4 - 3 = 1$.