Math 54: Quiz #3

February 23

GSI: M. Lindsey

Name: ________________________________

Please give neat and organized answers. Whenever applicable (especially for computational questions), make it clear what strategy you are using.

Problem 1

Let

$$A = \begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix}.$$

Find a basis Null(A) and a basis for Col(A).

Solution: Row-reduce A as follows

$$\begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix} \overset{R_2 \leftarrow R_2 + 2R_1}{\longrightarrow} \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & -2 & -10 & -6 \\ 0 & 1 & 5 & 7 \end{bmatrix}$$

$$\overset{R_2 \leftarrow -\frac{1}{2} R_2}{\longrightarrow} \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & 1 & 5 & 3 \\ 0 & 1 & 5 & 7 \end{bmatrix}$$

$$\overset{R_3 \leftarrow R_3 - R_2}{\longrightarrow} \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\overset{R_3 \leftarrow \frac{1}{4} R_3}{\longrightarrow} \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{R_1 \leftarrow R_1 + 3R_2}{\longrightarrow} \begin{bmatrix} 1 & 0 & 7 & 6 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{R_1 \leftarrow R_1 - 6R_3}{\longrightarrow} \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{R_2 \leftarrow R_2 - 3R_3}{\longrightarrow} \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The pivot columns of A are the first, second, and fourth columns, so these columns of A form a basis of Col(A), i.e.,

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 7 \end{pmatrix} \right\}$$

is a basis for Col(A).

Moreover, we can see that a general solution of $A\mathbf{x} = \mathbf{0}$ can be written in terms of the free variable x_3 as

$$\begin{pmatrix} -7x_3 \\ -5x_3 \\ x_3 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} -7 \\ -5 \\ 1 \\ 0 \end{pmatrix}.$$

Thus

$$\left\{ \begin{pmatrix} -7 \\ -5 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a basis for $\mathrm{Null}(A)$.

Problem 2

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that $\ker(T)$ is a subspace of $\mathbb{R}^n$.

Solution: We want to show (1) that $\ker(T)$ contains the zero vector, (2) that $\ker(T)$ is closed under scalar multiplication, and (3) that $\ker(T)$ is closed under addition.

(1) We know that $T(\mathbf{0}) = \mathbf{0}$ (since this is a general fact about linear transformations). [**Warning**: note that the $\mathbf{0}$ on the left-hand side of this equation is the zero vector in $\mathbb{R}^n$, while the $\mathbf{0}$ on the right-hand side is the zero vector in $\mathbb{R}^m$!] This means $\mathbf{0} \in \ker(T)$, as desired.

(2) Suppose that $\mathbf{v} \in \ker(T)$ and λ is a scalar. We then know that $T(\mathbf{v}) = \mathbf{0}$, and we want to show that $\lambda\mathbf{v} \in \ker(T)$, i.e., that $T(\lambda\mathbf{v}) = \mathbf{0}$. But $T(\lambda\mathbf{v}) = \lambda T(\mathbf{v}) = \lambda\mathbf{0} = \mathbf{0}$, as we wanted to show.

(3) Suppose that $\mathbf{v}, \mathbf{w} \in \ker(T)$. We then know that $T(\mathbf{v}) = \mathbf{0}$ and $T(\mathbf{w}) = \mathbf{0}$, and we want to show that $\mathbf{v} + \mathbf{w} \in \ker(T)$, i.e., that $T(\mathbf{v} + \mathbf{w}) = \mathbf{0}$. But $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w}) = \mathbf{0}$, as we wanted to show.