Math 54: Quiz #2 Solutions

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Please give neat and organized answers. Whenever applicable (especially for computational questions), make it clear what strategy you are using.

Problem 1

Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}.$$

Do the columns of A span $\mathbb{R}^3$?

Solution: The columns of A span $\mathbb{R}^3$ if and only if A has a pivot position in every row, i.e., if and only if any echelon form of A has no row of zeros. (This condition is also equivalent to the linear system $A\mathbf{x} = \mathbf{b}$ being consistent for every $\mathbf{b} \in \mathbb{R}^3$.)

Then row-reduce A as follows:

$$\begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + 3R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 5 & 3 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & -2 \end{bmatrix}.$$

Evidently we have a pivot position in every row, so the answer is ***yes***, the columns of A do span $\mathbb{R}^3$.

Problem 2

Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -4 \\ -2 \\ h \end{bmatrix}.$$

(a) For what values of h is $\mathbf{v}_3$ in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$?

Solution: Note that $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ if and only if the system

$$[\mathbf{v}_1 \ \mathbf{v}_2]\,\mathbf{x} = \mathbf{v}_3$$

is consistent. Then consider the augmented matrix

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 2 & 2 & -2 \\ 1 & 2 & h \end{bmatrix}$$

and row-reduce to echelon form:

$$\begin{bmatrix} 1 & 2 & -4 \\ 2 & 2 & -2 \\ 1 & 2 & h \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -4 \\ 0 & -2 & 6 \\ 1 & 2 & h \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \begin{bmatrix} 1 & 2 & -4 \\ 0 & -2 & 6 \\ 0 & 0 & h+4 \end{bmatrix}.$$

From this echelon form, we can tell that the last column is a pivot if and only if $h \neq -4$. Thus the system is consistent if and only if $h = -4$. Thus the only value of h for which $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is $h = -4$.

(b) For what values of h is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent?

Solution: Recall that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent if and only if the system

$$[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]\,\mathbf{x} = \mathbf{0}$$

has a nontrivial solution (i.e., if and only if the solution is not just the trivial solution $\mathbf{x} = 0$). To see whether this is the case, we should row-reduce the coefficient matrix $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$ to see if we have any free columns. Note that this matrix A is the same as in part (a), though the relevant interpretation of it is different in this part. Thus we have already done the work of row-reducing it, and we can see from above that A has a free column if and only if $h = -4$. Thus the only value of h for which $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent is $h = -4$.