Math 54: Linear independence and the Wronskian

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Consider n functions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ which take values in $\mathbb{R}^n$. The Wronskian is defined

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \det\left[\mathbf{x}_1(t)\ \mathbf{x}_2(t)\ \cdots\ \mathbf{x}_n(t)\right].$$

If $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are solutions of a homogeneous linear system of differential equations, i.e., of $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$, then **exactly one** of the following cases holds:

(1) $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = 0$ **for all** t, in which case $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are **linearly dependent**
(2) $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = 0$ **for NO values of** t, in which case $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are **linearly independent**

This implies a few things:

- If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ solve a homogeneous linear system of differential equations and you can find any value of t for which $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = 0$, then $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly dependent, hence **do not** form a fundamental solution set.
- If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ solve a homogeneous linear system of differential equations and you can find any value of t for which $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) \neq 0$, then $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent, hence form a fundamental solution set.
- If someone gives you some functions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and the corresponding Wronskian is zero for at least one value but not all values of t, then $\mathbf{x}_1, \ldots, \mathbf{x}_n$ **CANNOT** all be solutions of a single homogeneous linear system of differential equations.

Okay now let's consider what the Wronskian has to say when $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are not necessarily solutions of a homogeneous linear system of differential equations. The following fact holds:

- If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly dependent, then $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = 0$ **for all** t.

However, the converse **does not hold**. To see this, consider

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}.$$

Notice that the Wronksian is zero for all t, but $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent. Why? Suppose that there are constants c_1 and c_2 such that $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = \mathbf{0}$ for all t (in other words, such that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is the zero function). Then in fact $c_1 + c_2 t = 0$ for all t, but this means that $c_1 = c_2 = 0$, and we have proved linear independence. **Thus even though the vectors $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly dependent for every value of t, the functions $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent!**

We also have the following fact (the contrapositive of the last one):

- If $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) \neq 0$ **for some** t, then $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent.

In summary, the Wronskian is not a very reliable tool when your functions are **not** solutions of a homogeneous linear system of differential equations. However, if you find that the Wronskian is nonzero for some t, you **do** automatically know that the functions are linearly independent. (But the Wronskian being zero everywhere does not imply that the functions are linearly dependent, and linear independence does not imply that the Wronskian can't be the zero everywhere.)