Math 54: Fourier series

April 25

Let V be the set of piecewise continuous functions on $[-L, L]$. V is a vector space. We can make V into an inner product space by defining an inner product via

$$\langle f, g\rangle = \int_{-L}^{L} f(x)g(x)\,dx.$$

Define functions $f_0, f_1, g_1, f_2, g_2, \ldots$ in V via

$$f_0(x) = \frac{1}{\sqrt{2L}}$$

and

$$f_n(x) = \frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right), \quad g_n(x) = \frac{1}{\sqrt{L}}\sin\left(\frac{n\pi x}{L}\right)$$

for $n \geq 1$. Notice that these functions are periodic with respect to the interval $[-L, L]$.

In fact, $\{f_0, f_1, g_1, f_2, g_2, \ldots\}$ is an orthonormal set with respect to the inner product defined above. Even more amazingly, this set is an orthonormal 'basis' of sorts, so any (nice enough[1]) function can be written

$$f = \langle f, f_0\rangle f_0 + \sum_{n=1}^{\infty}\left[\langle f, f_n\rangle f_n + \langle g, g_n\rangle g_n\right].$$

Writing this out by plugging in our expressions for $f_0, f_1, g_1, f_2, g_2, \ldots$, we obtain

$$\begin{aligned}
f(x) &= \left(\int_{-L}^{L} f(y)\frac{1}{\sqrt{2L}}\,dy\right)\frac{1}{\sqrt{2L}} \\
&\quad + \sum_{n=1}^{\infty}\left(\int_{-L}^{L} f(y)\frac{1}{\sqrt{L}}\cos\left(\frac{n\pi y}{L}\right)\,dy\right)\frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right) \\
&\quad + \sum_{n=1}^{\infty}\left(\int_{-L}^{L} f(y)\frac{1}{\sqrt{L}}\sin\left(\frac{n\pi y}{L}\right)\,dy\right)\frac{1}{\sqrt{L}}\sin\left(\frac{n\pi x}{L}\right) \\
&= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(y)dy\right) \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f(y)\cos\left(\frac{n\pi y}{L}\right)\,dy\right)\cos\left(\frac{n\pi x}{L}\right) \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f(y)\sin\left(\frac{n\pi y}{L}\right)\,dy\right)\sin\left(\frac{n\pi x}{L}\right).
\end{aligned}$$

Therefore, if we define

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \text{ for } n = 0, 1, \ldots \quad\text{and}\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \text{ for } n = 1, 2, \ldots,}$$

then the Fourier series of f can be written

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right].}$$

This matches the conventions from the textbook (see page 576). (Notice that in fact $\langle f, f_n\rangle \neq a_n$, $\langle f, g_n\rangle \neq b_n$, so one must be careful about the normalization convention that one is using!)

[1] A considerable amount of extra math is needed to describe precisely what we mean by this.