COMPLEX NUMBERS

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Complex numbers. The set $\mathbb{C}$ of complex numbers is defined to be the set of $z = x + iy$ where x and y are real numbers (i.e., $x, y \in \mathbb{R}$). For such $z = x + iy$ we say that x is the *real part* and y is the *imaginary part*, and we write $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$. A general number $z \in \mathbb{C}$ is called *complex*, but if the real part of z is zero, i.e., if $\mathrm{Re}(z) = 0$, then we say that z is *imaginary* (or, for emphasis, *purely imaginary*). If the imaginary part of z is zero, i.e., if $\mathrm{Im}(z) = 0$, then we say that z is *real*.

$\mathbb{C}$ can be visualized as a plane. We view the real and imaginary parts x and y of a complex number $z = x + iy$ as the x- and y-coordinates of z in the *complex plane*. In the complex plane, the 'x-axis' is referred to as the *real axis*, and the 'y-axis' is referred to as the *imaginary axis*.

Complex addition. Complex numbers can be added as follows. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then the sum $z_1 + z_2$ is given by

$$z_1 + z_2 = (x_1 + y_1) + i(x_2 + y_2).$$

(If one identifies the complex plane $\mathbb{C}$ with $\mathbb{R}^2$, then the addition of complex numbers corresponds to usual vector addition in $\mathbb{R}^2$.)

Complex multiplication. $\mathbb{C}$ comes equipped with an extra structure that is not present in $\mathbb{R}^2$: two complex numbers can be multiplied to yield another complex number. Algebraic manipulation complex numbers is formally the same as the algebraic manipulation of numbers that you are used to, except with the extra relation that $i^2 = -1$. Complex multiplication is defined accordingly: if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2 y_1 y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2). \end{aligned}$$

In the last expression, we have written the complex number $z_1 z_2$ in a way that makes it clear what its real and imaginary parts are.

Note that complex multiplication (just like real multiplication) is commutative and associative (and satisfies the distributive property over addition).

Exponentiation of complex numbers. For $z \in \mathbb{C}$, it's not immediately clear what one wants the expression e^z to mean. One way to extend the definition of exponentiation to the complex numbers is via the Taylor series: simply define

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

Fortunately, we will not have to work with this definition directly! But note that, using this definition, one can guarantee that the usual rule

$$e^{z_1} e^{z_2} = e^{z_1 + z_2} \quad (\star)$$

holds for all $z_1, z_2 \in \mathbb{C}$. (Try proving this for a challenge!)

One can also use the Taylor series to describe the exponential of a purely imaginary number in terms of sines and cosines. (Note that we additionally have to recall the Taylor series for sine and cosine to carry out this proof; also try this for a challenge!) To wit, we have the following identity: if θ is a **real number**, then

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

This is called *Euler's formula*. (In particular, plugging in the value $\theta = \pi$ yields *Euler's identity*: $e^{i\pi} = -1$.)

Notice that Euler's formula, together with the rule $(\star)$, can be used to compute the real and imaginary parts of the exponential of an *arbitrary* complex number $z = x + iy$. Indeed, for such z,

$$e^z = e^{x+iy} = e^x e^{iy} = e^x \cos(y) + ie^x \sin(y),$$

so the real part of e^z is $e^x \cos(y)$ and the imaginary part is $e^x \sin(y)$. (In fact, we could have taken this as our definition of complex exponentiation, but the other definition is a bit more mathematically appealing.)

Polar form of a complex number. We can think of complex numbers in terms of their polar coordinates in the complex plane. What we mean by this is the following. Temporarily view $\mathbb{C}$ as the plane $\mathbb{R}^2$ (and forget about the extra structure). Then we can assign every complex number z polar coordinates (r, θ). These are defined such that $z = r\cos(\theta) + ir\sin(\theta)$. Note that the θ coordinate is really only defined up to integer multiples of 2π. We define the *modulus* (or *length* or *absolute value*), denoted $|z|$, of a complex number z to be its r-coordinate (i.e., its distance from the origin in the complex plane), and we define the *argument* (or *phase*), denoted $\arg(z)$, of z to be its θ-coordinate.

Complex multiplication can be understood in terms of these polar coordinates:

$$|z_1 z_2| = |z_1||z_2|, \qquad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi}.$$

In other words, multiplication by a complex number z induces a scaling by $|z|$ and a rotation by $\arg(z)$.

Since the r- and θ-coordinates of z were defined via $z = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta))$, we can use Euler's formula to conclude that

$$z = re^{i\theta}.$$

This is the *polar form* of the complex number z. (Importantly, in the polar form, both r and θ are real numbers; in fact $r \geq 0$.) By contrast, the form $z = x + iy$ is referred to as the *Cartesian form*. Note that the polar radius r can be written in terms of Cartesian coordinates as $r = \sqrt{x^2 + y^2}$.

In general, if we consider an arbitrary complex number $x = x + iy$, we might wonder how to write e^z in polar form. (Since z is not necessarily real, e^z is not already in polar form.) Then compute:

$$e^z = e^{x+iy} = e^x e^{iy}.$$

Note that $e^x > 0$ and $y \in \mathbb{R}$, so this is the polar form. In other words, the (r, θ) coordinates are (e^x, y).

Complex conjugation. If $z = x + iy$, then the *complex conjugate* $\overline{z}$ is defined by $\overline{z} = x - iy$. (The real part is the same as that of z, and the imaginary part is the negative of that of z.) In the complex plane, complex conjugation corresponds to reflection across the real axis.

Notice that $|\overline{z}| = |z|$, and $\arg(\overline{z}) = -\arg(z)$. Therefore, if we are given z in polar form, i.e., $z = re^{i\theta}$, then $\overline{z} = re^{-i\theta}$.

The following identity involving the complex conjugate holds: $z\overline{z} = |z|^2$. (In this equation, the left hand size is the product of *complex* numbers z and $\overline{z}$. So it's a special fact that the result, namely $|z|^2$, is in fact a real number.)

In fact, there are two ways to justify this identity: via the Cartesian representation and via polar representation. We will illustrate both. Using Cartesian coordinates, write $z = x + iy$ for $x, y \in \mathbb{R}$, and compute

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$

Alternatively, we can write z in polar form as $z = re^{i\theta}$, so $\overline{z} = re^{-i\theta}$, and

$$z\overline{z} = r^2 e^{i\theta - i\theta} = r^2 e^0 = r^2 = |z|^2.$$

We also have the following two identities:

$$\mathrm{Re}(z) = \frac{1}{2}\left(z + \overline{z}\right), \quad \mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z}).$$

To see the first, write $z = x + iy$, so

$$\frac{1}{2}(z + \overline{z}) = \frac{1}{2}(x + iy + x - iy) = \frac{1}{2}2x = x = \mathrm{Re}(z).$$

The second identity is established similarly:

$$\frac{1}{2i}(z - \overline{z}) = \frac{1}{2i}(x + iy - x + iy) = \frac{1}{2i}2iy = y = \mathrm{Im}(z).$$

Auxiliary equations with complex roots. When considering the ordinary differential equation (ODE)

$$ay''(t) + by'(t) + cy(t) = 0,$$

in the scenario that $b^2 - 4ac < 0$, the auxiliary equation

$$ar^2 + br + c = 0$$

has two complex roots r_1, r_2 which are complex conjugates of one another, i.e., $r_2 = \overline{r_1}$. Therefore we can write

$$r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta,$$

where α, β are real numbers.

We reasoned that $\tilde{y}_1(t) = e^{r_1 t}$ and $\tilde{y}_2(t) = e^{r_2 t}$ should form a basis of solutions of the ODE by analogy with the case in which $b^2 - 4ac > 0$ (where the roots r_1, r_2 were real). However, this is not an entirely satisfying solution because $\tilde{y}_1, \tilde{y}_2$ are complex-valued. Since our ODE involved only real numbers, we want to recover a basis of real-valued solutions.

Note that

$$\tilde{y}_1(t) = e^{r_1 t} = e^{\alpha t} e^{i\beta t} = e^{\alpha t}\left[\cos(\beta t) + i\sin(\beta t)\right]$$

and

$$\tilde{y}_2(t) = e^{r_2 t} = e^{\alpha t} e^{-i\beta t} = e^{\alpha t}\left[\cos(\beta t) - i\sin(\beta t)\right].$$

Then define

$$\begin{aligned} y_1(t) &:= \frac{1}{2}\left[\tilde{y}_1(t) + \tilde{y}_2(t)\right] = e^{\alpha t}\cos(\beta t) \\ y_2(t) &:= \frac{1}{2i}\left[\tilde{y}_1(t) - \tilde{y}_2(t)\right] = e^{\alpha t}\sin(\beta t). \end{aligned}$$

We see that $\{y_1, y_2\}$ is the *real-valued* basis of solutions that we were looking for.

Notice that actually

$$\tilde{y}_2(t) = \overline{\tilde{y}_1(t)},$$

and our definition of y_1, y_2 was equivalent to setting

$$y_1(t) := \mathrm{Re}\left(\tilde{y}_1(t)\right), \quad y_2(t) := \mathrm{Im}\left(\tilde{y}_1(t)\right).$$