Asymptotics of Hermite polynomials

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We motivate the study of the asymptotics of Hermite polynomials via their appearance in the analysis of the Gaussian Unitary Ensemble (GUE). Following Tao [3], we prove various facts about the Hermite polynomials and analyze their asymptotics via analysis of the semiclassical harmonic oscillator operator, which arises from the Hermite differential equation. Still following Tao [3], we use these asymptotics to prove a semicircular law for the expected empirical spectral distribution of the GUE. Since this project is a pedagogical exercise, we provide significantly more detail than Tao [3], filling in several computations and exercises left to the reader, and even correcting a few errors.

Lastly, we make use of our results on the asymptotics of Hermite polynomials to show that, in a certain sense, the normalized stationary states for the quantum harmonic oscillator converge to the classical position distribution for the harmonic oscillator. This convergence is suggested graphically in Griffiths [2]. Our proof is very likely nothing new, though it has been thought out independently by the author.

Before beginning, we note that our discussion of random matrix theory (focusing on the GUE) is less detailed than the material that follows (i.e., the discussion of Hermite polynomials, their asymptotics, and the proofs of the semicircular law and the fact about harmonic oscillator), and we refer to other texts for proofs in certain places. (Since this project is for a class on orthogonal polynomials, we decided that it would be more appropriate to provide full detail in the latter parts and to include the discussion of random matrix theory mainly as motivation.)

Contents

1 The GUE and related ensembles

The n-dimensional GUE is an ensemble of $n \times n$ Hermitian matrices. A sample M is obtained by sampling the upper triangular entries from the $\mathcal{N}_{\mathbb{C}}(0,1)$ distribution and sampling the diagonal entries from $\mathcal{N}_{\mathbb{R}}(0,1)$, all mutually independently, and fixing the lower-triangular entries by the constraint that M is Hermitian. We write $M = (M_{ij})$.

Consider the measure $dM = \prod_{i=1}^{n} dM_{ii} \prod_{j<k} dM_{jk}^{\Re} dM_{jk}^{\Im}$ on the space $\mathcal{M}$ of Hermitian matrices (simply the Lebesgue measure on $\mathcal{M}$ considered in the most obvious way as $\mathbb{R}^{n^2}$). It is clear by construction that the probability measure corresponding to the GUE is given by

$$\begin{aligned}
\mu_n &= C_n \left(\prod_{i=1}^{n} e^{-|M_{ii}|^2/2} dM_{ii} \right) \left(\prod_{1 \le j < k \le n} e^{-|M_{ij}|^2} dM_{jk}^{\Re} dM_{jk}^{\Im} \right) \\
&= C_n \left(\prod_{i=1}^{n} e^{-|M_{ii}|^2/2} \right) \left(\prod_{1 \le j < k \le n} e^{-|M_{ij}|^2} \right) dM \\
&= C_n e^{-\mathrm{tr}\left(M^2\right)/2} dM,
\end{aligned}$$

where C_n is chosen so that $\mu_n(\mathcal{M}) = 1$. It is clear that $e^{-\mathrm{tr}\left(M^2\right)/2}$ is invariant under the transformation $M \mapsto UMU^{-1}$ for unitary matrices U. By computing the determinant of $M \mapsto UMU^{-1}$ as a linear transformation $\mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ (as in Deift [2]), it can be shown that dM is also unvariant under this transformation, and therefore μ_n is invariant under conjugation by unitary matrices.

More generally we can consider the distribution $\mu_n^F = C_n^F e^{-F(M)} dM$, where $F : \mathcal{M} \to \mathbb{R}$ is invariant under unitary conjugation (so F depends symmetrically on all of the eigenvalues). Particularly noteworthy are the functions $F = \mathrm{tr} \circ P$, where P is a polynomial of even degree with positive coefficient for the highest-order term (ensuring integrability). Many of the properties of the GUE hold for these ensembles as well. For P quadratic, it is easy to see that $e^{-\mathrm{tr} P(M)}$ splits into a product of functions of M_{ii} and M_{jk} (with $j < k$) individually, so evidently the diagonal and upper triangular entries of M are all mutually independent in this case (as in the GUE). As mentioned by Deift [2], a suitable converse is true, and it follows that modulo shifting by a constant and rescaling, the GUE is the unique Hermitian matrix ensemble that is invariant under unitary conjugation with all non-lower-triangular entries mutually independent and therefore a very natural object to study.

2 The Ginibre formula

We consider M to be the random matrix variable associated with μ_n^F (though we shall omit the F from our notation) for $F = \mathrm{tr} \circ P$ as above. Consider the map λ which maps $M \mapsto (\lambda_1, \lambda_2, \cdots, \lambda_n)$ where $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ are the ordered eigenvalues of M (all real since M is hermitian). Then the following holds for the density ρ_n for λ on the space $\mathbb{R}^n_{\le} = \{x_1 \le x_2 \le \cdots \le x_n\}$:

Theorem. *(Ginibre formula).*

$$\rho_n(\lambda) = \text{const} \cdot e^{-\sum_{i=1}^n P(\lambda_i)} |\Delta_n(\lambda)|^2,$$

where

$$\Delta_n(\lambda) = \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i).$$

We give a brief sketch of the proof, following Deift [2]. First note that when the eigenvalues of M are simple, the diagonalization $M = U\Lambda U^*$ (with the entries of Λ ordered) is unique up to right-multiplication of U by diagonal unitary matrices (which form the maximal torus $\mathbb{T}^n \subset \mathcal{U}_n$). This holds because a matrix can only commute with Λ if it is diagonal. Also, it is not hard to see that generically M has simple eigenvalues (this can be shown for example by considering the vanishing set of the discriminant, which does not have full dimension). Then there is a map $M \overset{\phi}{\mapsto} (\mathbb{R}^n_< = \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}, \mathcal{U}_n \bmod (\mathbb{T}^n))$ which is a bijection when restricted a subset of $\mathcal{M}$ of full measure.

A calculation using local coordinates of $\mathcal{U}_n \bmod (\mathbb{T}^n)$ shows that the Jacobian determinant of ϕ is equal to the product of $\Delta_n(\lambda)$ and a factor that depends only on 'unitary variables' associated with the coordinates on $\mathcal{U}_n \bmod (\mathbb{T}^n)$, and with some work the Ginibre formula follows from this fact.

3 Enter orthogonal polynomials

The GUE and related ensembles are linked to orthogonal polynomials via the expression $\Delta_n(\lambda)$ that appears in the Ginibre formula. $\Delta_n(\lambda)$ is of course the Vandermonde determinant $\det V$ where $V = (\lambda_j^{i-1})$. For all $j \ge 0$, let π_j be a monic polynomial of degree j. Then by elementary row operations, we have that $\Delta_n(\lambda) = \det(V_\pi)$ where $V_\pi = \pi_i(\lambda_j)$, so

$$|\Delta_n(\lambda)|^2 = \det\left(V_\pi^\top V_\pi\right) = \det\left(\sum_{k=0}^{n-1} \pi_k(\lambda_i)\pi_k(\lambda_j)\right)_{1 \le i,j \le n}.$$

Then in the case of the GUE,

$$\begin{aligned}
\rho_n(\lambda) &= \text{const} \cdot e^{-\sum_{i=1}^n \lambda_i^2/2} \det\left(\sum_{k=0}^{n-1} \pi_k(\lambda_i)\pi_k(\lambda_j)\right)_{1 \le i,j \le n} \\
&= \text{const} \cdot \det\left(\sum_{k=0}^{n-1} \pi_k(\lambda_i) e^{-\lambda_i^2/4} \pi_k(\lambda_j) e^{-\lambda_j^2/4}\right)_{1 \le i,j \le n} \\
&= \text{const} \cdot \det\left(K_n(\lambda_i, \lambda_j)\right)_{1 \le i,j \le n},
\end{aligned}$$

where K_n is defined by $K_n(x,y) = \sum_{k=0}^{n-1} \pi_k(\lambda_i) e^{-\lambda_i^2/4} \pi_k(\lambda_j) e^{-\lambda_j^2/4}$.

We see that taking the π_j to be non-monic only changes the formula for $\rho_n(\lambda)$ by a constant factor, so let us take π_j to be the orthonormal Hermite polynomials. Then evidently by orthonormality we have that $\int K_n(x,x)\,dx = n$ (trace identity) and $K_n(x,y) = \int K_n(x,z)K_n(z,y)\,dz$ (reproducing formula). These two properties yield the following:

Lemma. *(Determinantal integration).*

$$\int \det\left(K_n(\lambda_i,\lambda_j)\right)_{1\le i,j\le k+1} d\lambda_{k+1} = (n-k)\int \det\left(K_n(\lambda_i,\lambda_j)\right)_{1\le i,j\le k}.$$

In particular it follows that

$$\int \det\left(K_n(\lambda_i,\lambda_j)\right)_{1\le i,j\le n} d\lambda_1\cdots d\lambda_n = n!$$

Proof. See, for example, Tao [3]. ☐

The latter statement can be used to show that in fact

$$\rho_n(\lambda) = \det\left(K_n(\lambda_i,\lambda_j)\right)_{1\le i,j\le n}$$

in the GUE case. Furthermore, it can be shown that $\rho_k(\lambda_1,\ldots,\lambda_k) := \det(K_n(\lambda_i,\lambda_j))_{1\le i,j\le k}$ is the k-point correlation function for the spectrum, i.e.,

$$\int_{\mathbb{R}^k} \rho_k(\lambda_1,\ldots,\lambda_k)F(\lambda_1,\ldots,\lambda_k)\,d\lambda_1\,\cdots\,d\lambda_k = \mathbf{E}\left[\sum_{1\le i_1,\ldots,i_k\le n,\,\text{distinct}} F\left(\lambda_{i_1}(M_n),\ldots,\lambda_{i_k}(M_n)\right)\right]$$

for any measurable $F:\mathbb{R}^k\to\mathbb{C}$ supported on $\{x_1\le\cdots\le x_k\}$. See Tao [3] for details.

It follows from the Christoffel-Darboux formula that

$$K_n(x,y) = \frac{\pi_n(x)\pi_{n-1}(y)-\pi_{n-1}(x)\pi_n(y)}{a_{n-1}(x-y)}e^{-(x^2+y^2)/4},$$

where a_i are defined by the three-term recurrence $\pi_{i+1}(x) = (a_i x + b_i)P_i(x) - c_i P_{i-1}(x)$. Then also we have that

$$K_n(x,x) = (a_{n-1})^{-1}\left(\pi_n'(x)\pi_{n-1}(x) - \pi_{n-1}'(x)\pi_n(x)\right)e^{-x^2/2}.$$

Then it is clear that pinning down the asymptotics of the π_n (and the a_n) will help us to grasp the asymptotic behavior of the expected spectral measure of the GUE.

Many objects associated with the GUE can be expressed conveniently in terms of K_n; see, for example, Deift [2]. We will later focus on a particular object: the expected empirical spectral measure of M_n. The empirical spectral measure μ_{M_n} is defined by $\mu_{M_n} = \sum_{i=1}^n \delta_{\lambda_i(M_n)}$. Recalling our result on the k-point correlation function above with $k=1$, we have that

$$\int_{\mathbb{R}} K_n(x,x)F(x)\,dx = \mathbf{E}\sum_{i=1}^n F\left(\lambda_i(M_n)\right)$$

for measurable $F:\mathbb{R}\to\mathbb{C}$. Of course,

$$\sum_{i=1}^n F\left(\lambda_i(M_n)\right) = \int F\sum_{i=1}^n \delta_{\lambda_i(M_n)},$$

so it follows that $\mathbf{E}\mu_{M_n} = K_n(x,x)\,dx$. As we shall confirm later, the range of the eigenvalues of M_n scales as $\sqrt{n}$, so for studying asymptotic behavior it makes sense to consider the rescaled probability measure $n^{-1/2}K_n(\sqrt{n}x,\sqrt{n}x)\,dx$, which is easily seen to be equal to $\mathbf{E}\mu_{M_n/\sqrt{n}}$. We write $E_n = n^{-1/2}K_n(\sqrt{n}x,\sqrt{n}x)$, so $\mathbf{E}\mu_{M_n/\sqrt{n}} = E_n(x)\,dx$. We will examine below the asymptotics of E_n and $\mathbf{E}\mu_{M_n/\sqrt{n}}$.

4 Hermite polynomials

Henceforth we denote by P_i the Hermite polynomials with positive leading coefficients which are orthonormal with respect to the weight function $w(x) = e^{-x^2/2}$.

4.1 The Hermite recurrence relation

By the three-term recurrence, there exist a_i, b_i, c_i (with $a_i \neq 0$, $c_0 = 0$) such that

$$P_{i+1}(x) = (a_i x + b_i)P_i(x) - c_i P_{i-1}(x).$$

We aim to compute the a_i, b_i, c_i. By taking the inner product of both sides with $P_{i+1}w$ and $P_{i-1}w$, we obtain

$$a_i^{-1} = \int xP_i(x)P_{i+1}(x)w(x)\,dx \tag{4.1}$$

$$c_i a_i^{-1} = \int xP_i(x)P_{i-1}(x)w(x)\,dx. \tag{4.2}$$

From (4.1) and (4.2) we see that $c_i = a_i a_{i-1}^{-1}$ for all $i \geq 1$.

Meanwhile, taking the inner product of both sides of the recurrence relation with $P_i w$ yields

$$b_i = -a_i \int xP_i(x)^2 w(x)\,dx. \tag{4.3}$$

(Thus far we have used only the orthonormality of the P_i.)

Notice that $xw(x) = -w'(x)$. Thus we obtain by integrating by parts in (4.1):

$$\begin{aligned} a_i^{-1} &= \int \left[P_i'(x)P_{i+1}(x) + P_i(x)P_{i+1}'(x)\right] w(x)\,dx \\ &= \int P_i(x)P_{i+1}'(x)w(x)\,dx, \end{aligned}$$

where the second equality follows from the fact that P_i' has degree at most $i-1$ (and hence is orthogonal to P_{i+1} with respect to w).

Let α_i be the leading coefficient of P_i. Then $P_{i+1}' = (i+1)\frac{\alpha_{i+1}}{\alpha_i}P_i + Q$, where Q is a polynomial of order at most $i-1$. Then

$$a_i^{-1} = \int P_i(x)\left[(i+1)\frac{\alpha_{i+1}}{\alpha_i}P_i(x) + Q(x)\right] w(x)\,dx = (i+1)\frac{\alpha_{i+1}}{\alpha_i}.$$

It is also clear from the three-term recurrence relation that $\alpha_{i+1} = a_i\alpha_i$, so $a_i^{-1} = (i+1)a_i$, i.e.,

$$a_i = \frac{1}{\sqrt{i+1}}, \ \ \alpha_{i+1} = \frac{\alpha_i}{\sqrt{i+1}}.$$

It is easy to see that $P_0(x) = (2\pi)^{-1/4}$, so

$$\alpha_i = \frac{1}{2\pi^{1/4}\sqrt{i!}}.$$

Also, since $c_i = a_i a_{i-1}^{-1}$ for $i \geq 1$, we have that

$$c_i = \frac{\sqrt{i}}{\sqrt{i+1}}$$

for $i \geq 1$.

Now integrating (4.3) by parts, we have

$$b_i = -2a_i \int P_i(x)P_i'(x)w(x)\,dx = 0,$$

where the second equality follows from the fact that P_i' has degree at most $i-1$.

Then the Hermite recurrence relation is

$$P_{i+1}(x) = \frac{1}{\sqrt{i+1}} x P_i(x) - \frac{\sqrt{i}}{\sqrt{i+1}} P_{i-1}(x). \tag{4.4}$$

4.2 Differential equations associated with Hermite polynomials

Next we obtain differential equations satisfied by the Hermite polynomials. To do so, we write P_{i+1}' as a linear combination of Hermite polynomials of order at most i, obtaining the coefficients of this linear combination by taking inner products. Note that by integration by parts we have

$$\begin{aligned}\int P_j(x)P_{i+1}'(x)w(x)\,dx &= \int xP_j(x)P_{i+1}(x)w(x)\,dx - \int P_j'(x)P_{i+1}(x)w(x)\,dx \\ &= \int \left[xP_j(x) - P_j'(x)\right] P_{i+1}(x)w(x)\,dx.\end{aligned}$$

When $j < i$, clearly the last integral is zero, and when $j = i$, we have that the contribution of the P_j' term is zero, so

$$\int P_i(x)P_{i+1}'(x)w(x)\,dx = \int xP_i(x)P_{i+1}(x)w(x)\,dx = a_i^{-1},$$

where the second equality follows from (4.1). Thus we can write

$$P_{i+1}' = a_i^{-1}P_i = \sqrt{i+1}P_i. \tag{4.5}$$

Then it follows that

$$(wP_i)'(x) = w'(x)P_i(x) + w(x)P_i'(x) = \left[-xP_i(x) + \sqrt{i}P_{i-1}\right] w(x).$$

By (4.4), the last expression is equal to $-\sqrt{i+1}P_{i+1}(x)w(x)$, so we have that

$$(wP_i)' = -\sqrt{i+1}wP_{i+1},$$

or perhaps more familiarly,

$$\frac{d}{dx}\left(e^{-x^2/2}P_i(x)\right) = -\sqrt{i+1}e^{-x^2/2}P_{i+1}(x).$$

Since $P_0 \equiv (2\pi)^{-1/4}$, we have that

$$P_n(x) = \frac{(-1)^n}{(2\pi)^{1/4}\sqrt{n!}}e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}. \tag{4.6}$$

This gives an explicit formula for the Hermite polynomials. However, we are interested in their asymptotic behavior for large n, which is not obvious from this formula.

We need another approach to uncover this asymptotic behavior. Note that by (4.4) and (4.5), we have for $n \geq 2$ that

$$\begin{aligned} nP_n(x) &= x\sqrt{n}P_{n-1}(x) - \sqrt{n}\sqrt{n-1}P_{n-2}(x) \\ &= xP_n'(x) - P_n''(x). \end{aligned}$$

Then we have the Hermite differential equation:

$$P_n''(x) - xP_n'(x) + nP_n(x) = 0.$$

We can in fact derive a differential equation satsified by the n-th Hermite function $\phi_n = P_n w^{1/2}$. We simply differentiate twice:

$$\phi_n'' = P_n'' w^{1/2} + 2P_n'\left(w^{1/2}\right)' + P_n\left(w^{1/2}\right)''.$$

Noting that $\left(w^{1/2}\right)'(x) = -\frac{x}{2}w(x)^{1/2}$, so therefore

$$\left(w^{1/2}\right)''(x) = \left(\frac{x^2}{4} - \frac{1}{2}\right) w(x)^{1/2},$$

we obtain

$$\phi_n''(x) = P_n''(x)w(x)^{1/2} - xP_n'(x)w(x)^{1/2} + \left(\frac{x^2}{4} - \frac{1}{2}\right) P_n(x)w(x)^{1/2}$$

$$
\begin{aligned}
&= \left[P_n''(x) - xP_n'(x) + \left(\frac{x^2}{4} - \frac{1}{2}\right) P_n(x)\right] w(x)^{1/2} \\
&= \left[-nP_n(x) + \left(\frac{x^2}{4} - \frac{1}{2}\right) P_n(x)\right] w(x)^{1/2} \\
&= \left[-\left(n + \frac{1}{2}\right) + \frac{x^2}{4}\right] \phi_n(x),
\end{aligned}
$$

i.e.,

$$
\boxed{\phi_n''(x) = -\left[\left(n + \frac{1}{2}\right) - \frac{x^2}{4}\right] \phi_n(x). \qquad (4.7)}
$$

Note that, equivalently, we have that

$$
L\phi_n = \left(n + \frac{1}{2}\right) \phi_n,
$$

where we denote by L the harmonic oscillator operator

$$
L\phi(x) = -\phi''(x) + \frac{x^2}{4}\phi.
$$

4.3 Rewriting K_n in terms of Hermite functions

Recall from above that (following from the Christoffel-Darboux formula)

$$
K_n(x,x) = a_{n-1}^{-1} \left(P_n'(x)P_{n-1}(x) - P_{n-1}'(x)P_n(x)\right) w(x).
$$

Since $P_{n-1} = a_{n-1}P_n'$ (see (4.5)), it follows that

$$
K_n(x,x) = \left(P_n'(x)^2 - P_n''(x)P_n(x)\right) w(x). \tag{4.8}
$$

Now

$$
\phi_n'(x) = P_n'(x)w(x)^{1/2} - \frac{x}{2}P_n(x)w(x)^{1/2},
$$

so

$$
\phi_n'(x)^2 = P_n'(x)^2 w(x) - xP_n(x)P_n'(x)w(x) + \frac{x^2}{4}P_n(x)^2 w(x)
$$

Then

$$
\begin{aligned}
\phi_n'(x)^2 + \left(n - \frac{x^2}{4}\right)\phi_n(x)^2 &= P_n'(x)^2 w(x) - xP_n(x)P_n'(x)w(x) + nP_n(x)^2 w(x) \\
&= P_n'(x)^2 w(x) + \left[-xP_n'(x) + nP_n(x)\right] P_n(x)w(x) \\
&= P_n'(x)^2 w(x) + \left[-x\sqrt{n}P_{n-1}(x) + nP_n(x)\right] P_n(x)w(x),
\end{aligned}
$$

where the last equality follows from (4.5). By the Hermite recurrence relation (4.4),

$$-x\sqrt{n}P_{n-1}(x) + nP_n(x) = -\sqrt{n(n-1)}P_{n-2}(x),$$

and again by (4.5), the RHS of this equation is equal to $P_n''(x)$, so

$$\phi_n'(x)^2 + \left(n - \frac{x^2}{4}\right)\phi_n(x)^2 = \left(P_n'(x)^2 - P_n''(x)P_n(x)\right)w(x).$$

Then by (4.8), we have that

$$K_n(x,x) = \phi_n'(x)^2 + \left(n - \frac{x^2}{4}\right)\phi_n(x)^2.$$

Recall from above that we were interested in the asymptotic behavior of $E_n = n^{-1/2}K_n(\sqrt{n}x, \sqrt{n}x)$, since the rescaled expected empirical spectral measure of the GUE is given by $E_n\,dx$. Now

$$\begin{aligned} E_n &= n^{-1/2}\phi_n'(\sqrt{n}x)^2 + \sqrt{n}\left(1 - \frac{x^2}{4}\right)\phi_n(\sqrt{n}x)^2 \\ &= n^{-2}\psi_n'(x)^2 + \left(1 - \frac{x^2}{4}\right)\psi_n(x)^2, \end{aligned}$$

where ψ_n is the rescaled Hermite function defined by $\psi_n(x) := n^{1/4}\phi_n(\sqrt{n}x)$.

Notice that ψ_n satisfies the differential equation

$$\begin{aligned} -n^{-2}\psi_n''(x) + \frac{x^2}{4}\psi_n(x) &= -n^{-3/4}\phi_n''(\sqrt{n}x) + n^{1/4}\frac{x^2}{4}\phi_n(\sqrt{n}x) \\ &= -n^{1/4}n^{-1}\left[-\left(n + \frac{1}{2}\right) + n\frac{x^2}{4}\right]\phi_n(\sqrt{n}x) + n^{1/4}\frac{x^2}{4}\phi_n(\sqrt{n}x) \\ &= \left[\frac{n+\frac{1}{2}}{n} - \frac{x^2}{4}\right]n^{1/4}\phi_n(\sqrt{n}x) + \frac{x^2}{4}n^{1/4}\phi_n(\sqrt{n}x) \\ &= \left(1 + \frac{1}{2n}\right)\psi_n, \end{aligned}$$

i.e., $L_{1/n}\psi_n = \left(1 + \frac{1}{2n}\right)\psi_n$, where L_h denotes the semiclassical harmonic oscillator operator

$$L_h\psi = -h^2\psi'' + \frac{x^2}{4}\psi.$$

Then it is natural to analyze the eigenfunction equation

$$L_h\psi = \lambda\psi,$$

or equivalently,

$$\psi'' = -\left(\lambda - \frac{x^2}{4}\right)\psi = -\frac{1}{h^2}k(x)^2\psi, \tag{4.9}$$

where $k(x) := \sqrt{\lambda - x^2/4}$ (so we consider only the region where $x^2/4 < \lambda$). We will eventually take $h > 0$ to be $1/n$ (small when n is large) and λ to be $1 + \frac{1}{2n}$ (nearly 1 when n is large).

5 Analysis of the eigenfunction equation of L_h

It is helpful, both intuitively and formally, to draw a comparison between (4.9) and the familiar ODE $\psi'' = -h^{-2}k^2\psi$ with constant coefficient, having as its solution $\psi(x) = Ae^{ikx/h} + Be^{-ikx/h}$. We then forward the ansatz

$$\psi(x) = A(x)e^{i\Psi(x)/h} + B(x)e^{-i\Psi(x)/h},$$

where $\Psi(x) = \int_0^x k(y)\,dy$, i.e., Ψ is an antiderivative of k. We obtain by differentiating:

$$\psi'(x) = \frac{ik(x)}{h}\left(A(x)e^{i\Psi(x)/h} - B(x)e^{-i\Psi(x)/h}\right) + A'(x)e^{i\Psi(x)/h} + B'(x)e^{-i\Psi(x)/h}.$$

Since we are 'solving' for two functions A and B, heuristically we expect to be able to put an additional constraint on A and B, namely

$$A'(x)e^{i\Psi(x)/h} + B'(x)e^{-i\Psi(x)/h} = 0. \tag{5.1}$$

We will see later that the expressions for A and B that we obtain do indeed satisfy this equation.

Again we differentiate, yielding

$$\begin{aligned}\psi''(x) &= -\frac{k(x)^2}{h^2}\psi(x) + \frac{ik'(x)}{h}\left(A(x)e^{i\Psi(x)/h} - B(x)e^{-i\Psi(x)/h}\right)\\ &\quad + \frac{ik(x)}{h}\left(A'(x)e^{i\Psi(x)/h} - B'(x)e^{-i\Psi(x)/h}\right).\end{aligned}$$

Referring back to (4.9), we see that then

$$A'(x)e^{i\Psi(x)/h} - B'(x)e^{-i\Psi(x)/h} = -\frac{k'(x)}{k(x)}\left(A(x)e^{i\Psi(x)/h} - B(x)e^{-i\Psi(x)/h}\right).$$

Adding and subtracting this equation with (5.1), we obtain

$$\begin{aligned}2A'(x)e^{i\Psi(x)/h} &= -\frac{k'(x)}{k(x)}\left(A(x)e^{i\Psi(x)/h} - B(x)e^{-i\Psi(x)/h}\right)\\ 2B'(x)e^{-i\Psi(x)/h} &= \frac{k'(x)}{k(x)}\left(A(x)e^{i\Psi(x)/h} - B(x)e^{-i\Psi(x)/h}\right),\end{aligned}$$

or equivalently

$$\begin{aligned}A'(x) &= -\frac{k'(x)}{2k(x)}A(x) + \frac{k'(x)}{2k(x)}B(x)e^{-2i\Psi(x)/h}\\ B'(x) &= -\frac{k'(x)}{2k(x)}B(x) + \frac{k'(x)}{2k(x)}A(x)e^{2i\Psi(x)/h}.\end{aligned}$$

Define $a(x) = k(x)^{1/2}A(x)$ and $b(x) = k(x)^{1/2}B(x)$, so

$$A(x) = k(x)^{-1/2}a(x), \quad B(x) = k(x)^{-1/2}b(x),$$

and also

$$\begin{aligned} a'(x) &= k(x)^{1/2}A'(x) + \frac{k'(x)}{2k(x)^{1/2}}A(x) \\ &= \frac{k'(x)}{2k(x)^{1/2}}B(x)e^{-2i\Psi(x)/h} \\ &= \frac{k'(x)}{2k(x)}b(x)e^{-2i\Psi(x)/h}, \end{aligned}$$

i.e.,

$$a'(x) = \frac{k'(x)}{2k(x)}b(x)e^{-2i\Psi(x)/h}, \tag{5.2}$$

and similarly

$$b'(x) = \frac{k'(x)}{2k(x)}a(x)e^{2i\Psi(x)/h}.$$

Then on any compact interval I contained within $(-2, 2)$, we have that $a', b' = O(|a|, |b|)$. Furthermore, by Gronwall's inequality we have that $|a(x)| \leq C_1|b(0)|$ and $|b(x)| \leq C_2|a(0)|$ on I for constants C_1, C_2 depending only on I. Therefore

$$a', b', a, b = O(|a(0)| + |b(0)|)$$

on I.

Now by (5.2), we have

$$\begin{aligned} a(x) - a(0) &= \int_0^x a'(u)\,du \\ &= \int_0^x \frac{k'(u)}{2k(u)}b(u)e^{-2i\Psi(u)/h}\,du \\ &= \int_0^{\Psi(x)} \frac{k'\left(\Psi^{-1}(v)\right)}{2k\left(\Psi^{-1}(v)\right)}\left(\Psi^{-1}\right)'(v)b\left(\Psi^{-1}(v)\right)e^{-2iv/h}\,dv, \end{aligned}$$

where we have made the substitution $v = \Psi(u)$ (note that Ψ is strictly increasing, hence invertible, since k is positive). Then integrating by parts we obtain

$$a(x) - a(0) = \frac{-h}{2i}\left(\left[\frac{k'\left(\Psi^{-1}(v)\right)}{2k\left(\Psi^{-1}(v)\right)}\left(\Psi^{-1}\right)'(v)b\left(\Psi^{-1}(v)\right)e^{-2iv/h}\right]_{v=0}^{v=\Psi(x)} - \int_0^{\Psi(x)} F(v)e^{-2iv/h}\,dv\right),$$

where

$$F(v) = \frac{d}{dv}\left[\frac{k'\left(\Psi^{-1}(v)\right)}{2k\left(\Psi^{-1}(v)\right)}\left(\Psi^{-1}\right)'(v)b\left(\Psi^{-1}(v)\right)\right].$$

In particular is a smooth function on I that does not depend on h, and furthermore from the product rule it is clear that $F = O(|b| + |b'|)$. It follows from our earlier finding that $F = O(|b(0)| + |b'(0)|)$. Evidently then

$$a(x) = a(0) + O\left(h\left[|a(0)| + |b(0)|\right]\right),$$

and by analogous reasoning we obtain

$$b(x) = b(0) + O\left(h\left[|a(0)| + |b(0)|\right]\right).$$

We have thus shown the following:

Theorem. *Suppose that $L_h\psi = \lambda\psi$. Then on any compact interval contained in $(-2, 2)$,*

$$\psi(x) = k(x)^{-1/2}\left(a(x)e^{i\Psi(x)/h} + b(x)e^{-i\Psi(x)/h}\right),$$

where

$$a(x) = a(0) + O\left(h\left[|a(0)| + |b(0)|\right]\right), \quad b(x) = b(0) + O\left(h\left[|a(0)| + |b(0)|\right]\right).$$

Furthermore

$$\psi'(x) = \frac{ik(x)^{1/2}}{h}\left(a(x)e^{i\Psi(x)/h} - b(x)e^{-i\Psi(x)/h}\right).$$

6 Asymptotics for the GUE expected empirical spectral measure

Now plugging in $\lambda = 1 + \frac{1}{2n}$ and $h = \frac{1}{n}$, we see that on any compact interval I contained in $(-2, 2)$,

$$\psi_n = k(x)^{-1/2}\left(a(x)e^{in\Psi(x)} + b(x)e^{-in\Psi(x)}\right),$$

where

$$a(x) = a(0) + O\left(n^{-1}\left[|a(0)| + |b(0)|\right]\right), \quad b(x) = b(0) + O\left(n^{-1}\left[|a(0)| + |b(0)|\right]\right),$$

and furthermore

$$\psi_n'(x) = nik(x)^{1/2}\left(a(x)e^{in\Psi(x)} - b(x)e^{-in\Psi(x)}\right).$$

In the following the reader should note that a and b depend on n, though we omit this dependence from the notation. Recall from above that the density of the expected empirical spectral measure is given by

$$E_n(x) = \left(1 - \frac{x^2}{4}\right)|\psi_n(x)|^2 + n^{-2}|\psi_n'(x)|^2,$$

so

$$\begin{aligned} E_n(x) &= \left|a(x)e^{in\Psi(x)} + b(x)e^{-in\Psi(x)}\right|^2 k(x)^{-1}\left(1 - \frac{x^2}{4}\right) \\ &\quad + \left|a(x)e^{in\Psi(x)} - b(x)e^{-in\Psi(x)}\right|^2 k(x) \\ &= \left[|a(x)|^2 + |b(x)|^2 + 2\Re\left(a(x)b(x)e^{2in\Psi(x)}\right)\right] k(x)^{-1}\left(1 - \frac{x^2}{4}\right) \\ &\quad + \left[|a(x)|^2 + |b(x)|^2 - 2\Re\left(a(x)b(x)e^{2in\Psi(x)}\right)\right] k(x) \\ &= \left[|a(x)|^2 + |b(x)|^2\right]\left[k(x)^{-1}\left(1 - \frac{x^2}{4}\right) + k(x)\right] \end{aligned}$$

$$+ 2\Re\left(a(x)b(x)e^{2in\Psi(x)}\right)\left[k(x)^{-1}\left(1-\frac{x^2}{4}\right)-k(x)\right]. \tag{6.1}$$

Notice that $k(x) = \sqrt{1+\frac{1}{2n}-\frac{x^2}{4}}$, so $k(x)^{-1}\left(1-\frac{x^2}{4}\right) \approx \sqrt{1-\frac{x^2}{4}}$ for large n. We will make this notion more precise. Indeed, we have that

$$\begin{aligned}
\left[k(x)^{-1}\left(1-\frac{x^2}{4}\right)\right]^2 &= \frac{\left(1-\frac{x^2}{4}\right)^2}{1+\frac{1}{2n}-\frac{x^2}{4}} \\
&= \frac{\left(1+\frac{1}{2n}-\frac{x^2}{4}\right)^2-\frac{1}{n}\left(1-\frac{x^2}{4}\right)-\frac{1}{4n^2}}{1+\frac{1}{2n}-\frac{x^2}{4}} \\
&= \left(1+\frac{1}{2n}-\frac{x^2}{4}\right)-\frac{1}{n}\cdot\frac{\left(1-\frac{x^2}{4}\right)+\frac{1}{4n}}{1+\frac{1}{2n}-\frac{x^2}{4}} \\
&= 1-\frac{x^2}{4}+O\left(\frac{1}{n}\right).
\end{aligned}$$

Notice that for $\alpha \geq \beta > 0$, we have that $\alpha - \beta \leq \alpha + \beta$, and $(\alpha-\beta)^2 \leq \alpha^2 - \beta^2$. Thus in fact for all $\alpha, \beta > 0$ we have that $(\alpha-\beta)^2 \leq |\alpha^2-\beta^2|$, so

$$\left[k(x)^{-1}\left(1-\frac{x^2}{4}\right)-\sqrt{1-\frac{x^2}{4}}\right]^2 = O\left(\frac{1}{n}\right),$$

and

$$k(x)^{-1}\left(1-\frac{x^2}{4}\right) = \sqrt{1-\frac{x^2}{4}}+o(1).$$

Similarly, it is easy to show that $k(x) = \sqrt{1-\frac{x^2}{4}}+o(1)$. Then we have from (6.1) that

$$E_n(x) = \left[|a(x)|^2+|b(x)|^2\right]\left[\sqrt{4-x^2}+o(1)\right]+O(|a(x)||b(x)|)\cdot o(1).$$

Of course since $a, b = O(|a(0)|, |b(0)|)$, we in fact have that

$$E_n(x) = \left[|a(x)|^2+|b(x)|^2\right]\left[\sqrt{4-x^2}+o(1)\right]+O(|a(0)||b(0)|)\cdot o(1). \tag{6.2}$$

To proceed we approximate the 'initial conditions' $a(0)$ and $b(0)$. Recall (4.6) froma above, i.e., that

$$P_n(x) = \frac{(-1)^n}{(2\pi)^{1/4}\sqrt{n!}}e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}.$$

Using the Taylor expansion

$$e^{-x^2/2} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^k k!},$$

we see that for even n,

$$P_n(0) = \frac{(-1)^{n/2}\sqrt{n!}}{(2\pi)^{1/4}2^{n/2}(n/2)!}, \quad P_n'(0) = 0,$$

and for odd n,

$$P_n(0) = 0, \quad P_n'(0) = \frac{(-1)^{(n+1)/2}(n+1)\sqrt{n!}}{(2\pi)^{1/4}2^{(n+1)/2}((n+1)/2)!}.$$

Now $\psi_n(0) = n^{1/4}P_n(0)$ and $\psi_n'(0) = n^{3/4}\phi_n'(0) = n^{3/4}P_n'(0)$, and furthermore $\psi_n(0) = a(0) + b(0)$ and $\psi_n'(0) = ni(a(0) - b(0))$, so

$$\begin{cases} a(0) + b(0) = \frac{(-1)^{n/2}n^{1/4}\sqrt{n!}}{(2\pi)^{1/4}2^{n/2}(n/2)!}, \quad a(0) - b(0) = 0, & n \text{ is even} \\ a(0) + b(0) = 0, \quad a(0) - b(0) = \frac{(-1)^{(n+1)/2}n^{-1/4}(n+1)\sqrt{n!}}{i(2\pi)^{1/4}2^{(n+1)/2}((n+1)/2)!}, & n \text{ is odd.} \end{cases}$$

Now recalling that $n! \sim \sqrt{2\pi n}\,(n/e)^n$ (Stirling's approximation), we have that

$$\begin{aligned} \frac{(-1)^{n/2}n^{1/4}\sqrt{n!}}{(2\pi)^{1/4}2^{n/2}(n/2)!} &\sim (-1)^{n/2}\frac{n^{1/4}(2\pi)^{1/4}n^{1/4}n^{n/2}e^{-n/2}}{(2\pi)^{1/4}2^{n/2}(n/2)!} \\ &= (-1)^{n/2}\frac{n^{1/2}n^{n/2}e^{-n/2}}{2^{n/2}(n/2)!} \\ &\sim (-1)^{n/2}\frac{n^{1/2}n^{n/2}e^{-n/2}}{2^{n/2}\pi^{1/2}n^{1/2}n^{n/2}2^{-n/2}e^{-n/2}} \\ &= (-1)^{n/2}\pi^{-1/2}, \end{aligned}$$

so

$$\frac{n^{1/4}\sqrt{n!}}{(2\pi)^{1/4}2^{n/2}(n/2)!} \to \pi^{-1/2}$$

as $n \to \infty$. We also compute

$$\begin{aligned} \frac{(-1)^{(n+1)/2}n^{-1/4}(n+1)\sqrt{n!}}{(2\pi)^{1/4}2^{(n+1)/2}((n+1)/2)!} &\sim (-1)^{(n+1)/2}\frac{n^{-1/4}(n+1)(2\pi)^{1/4}n^{1/4}n^{n/2}e^{-n/2}}{(2\pi)^{1/4}2^{(n+1)/2}((n+1)/2)!} \\ &= (-1)^{(n+1)/2}\frac{(n+1)n^{n/2}e^{-n/2}}{2^{(n+1)/2}((n+1)/2)!} \\ &= (-1)^{(n+1)/2}\frac{(n+1)n^{n/2}e^{-n/2}}{2^{(n+1)/2}\pi^{1/2}(n+1)^{1/2}(n+1)^{(n+1)/2}(2e)^{-(n+1)/2}} \\ &= (-1)^{(n+1)/2}\pi^{-1/2}e^{1/2}\left(\frac{n}{n+1}\right)^{n/2} \\ &= (-1)^{(n+1)/2}\pi^{-1/2}e^{1/2}\left(\left(1+\frac{1}{n}\right)^n\right)^{-1/2} \\ &\sim (-1)^{(n+1)/2}\pi^{-1/2}, \end{aligned}$$

so

$$\frac{n^{-1/4}(n+1)\sqrt{n!}}{(2\pi)^{1/4}2^{(n+1)/2}((n+1)/2)!} \to \pi^{-1/2}.$$

Now when n is even, $a(0) = b(0)$, so $a(0) = b(0) = \frac{1}{2}(a(0) + b(0))$. When n is odd, $a(0) = -b(0)$, so $a(0) = -b(0) = \frac{1}{2}(a(0) - b(0))$. It follows that we have that $|a(0)|, |b(0)| \to \frac{1}{2}\pi^{-1/2}$ as $n \to \infty$, so $|a(0)|^2 + |b(0)|^2 \to \frac{1}{2\pi}$ as $n \to \infty$.

Recall (6.2) from above, i.e., that

$$E_n(x) = \left[|a(x)|^2 + |b(x)|^2\right]\left[\sqrt{4 - x^2} + o(1)\right] + O(|a(0)||b(0)|) \cdot o(1).$$

Also recall that

$$a(x) = a(0) + O\left(n^{-1}\left[|a(0)| + |b(0)|\right]\right), \quad b(x) = b(0) + O\left(n^{-1}\left[|a(0)| + |b(0)|\right]\right).$$

But since $|a(0)|, |b(0)| \to \frac{1}{2}\pi^{-1/2}$, so $|a(0)| + |b(0)|$ and $|a(0)||b(0)|$ are both bounded uniformly in n, and in fact

$$a(x) = a(0) + O(n^{-1}), \quad b(x) = b(0) + O(n^{-1}),$$

and $O(|a(0)||b(0)|) = O(1)$. Thus $\left(|a|^2 + |b|^2\right) - \left(|a(0)|^2 + |b(0)|^2\right) \to 0$ uniformly on I as $n \to \infty$, so $|a|^2 + |b|^2 \to \frac{1}{2\pi}$ uniformly on I as $n \to \infty$, i.e., $|a|^2 + |b|^2 = \frac{1}{2\pi} + o(1)$. Then we have shown that

$$\begin{aligned} E_n(x) &= \left[\frac{1}{2\pi} + o(1)\right]\left[\sqrt{4 - x^2} + o(1)\right] + o(1) \\ &= \frac{1}{2\pi}\sqrt{4 - x^2} + o(1), \end{aligned}$$

or, in other words, $E_n(x) - \frac{1}{2\pi}\sqrt{4 - x^2}$ converges to zero uniformly on I.

Now let $\xi : \mathbb{R} \to \mathbb{R}$ be measurable, bounded, and supported on a compact interval $I \subset (-2, 2)$. Now $E_n(x) - \frac{1}{2\pi}\sqrt{4 - x^2} \to 0$ uniformly on I, so

$$\int_{-2}^{2} |\xi(x)| \left| E_n(x) - \frac{1}{2\pi}\sqrt{4 - x^2} \right| dx \to 0$$

by bounded convergence, and consequently $\int_{-2}^{2} \xi E_n \to \frac{1}{2\pi}\int_{-2}^{2} \xi(x)\sqrt{4 - x^2}\, dx$.

In particular, letting $I_m = [-2 + m^{-1}, 2 - m^{-1}]$ and taking $\xi \equiv \chi_{I_m}$ we see that $\int_{I_m} E_n \to \frac{1}{2\pi}\int_{I_m} \sqrt{4 - x^2}\, dx$. Of course, $\int_{\mathbb{R}} E_n = 1$, so it follows that $\int_{\mathbb{R}\setminus I_m} E_n \to 1 - \frac{1}{2\pi}\int_{I_m} \sqrt{4 - x^2}\, dx$.

Now let $\xi : \mathbb{R} \to \mathbb{R}$ be bounded and continuous, so $|\xi| \le B$ everywhere for some $B > 0$. Then

$$\begin{aligned} \int |\xi(x)| \left| E_n(x) - \frac{1}{2\pi}\sqrt{(4 - x^2)_+} \right| dx \quad \le \quad & \int_{I_m} |\xi(x)| \left| E_n(x) - \frac{1}{2\pi}\sqrt{4 - x^2} \right| dx \\ & + \int_{\mathbb{R}\setminus I_m} |\xi(x)| E_n(x)\, dx \\ & + \int_{\mathbb{R}\setminus I_m} |\xi(x)| \frac{1}{2\pi}\sqrt{(4 - x^2)_+}\, dx. \end{aligned}$$

The first term in the last expression converges to zero by our previous argument, and the second term is bounded by $B \int_{\mathbb{R}\setminus I_m} E_n(x)\,dx$, which converges to $B\left(1 - \frac{1}{2\pi}\int_{I_m}\sqrt{4-x^2}\,dx\right)$. Thus

$$\limsup_{n\to\infty} \int |\xi(x)| \left| E_n(x) - \frac{1}{2\pi}\sqrt{(4-x^2)_+} \right| dx \le 2B\left(1 - \frac{1}{2\pi}\int_{I_m}\sqrt{4-x^2}\,dx\right).$$

The RHS of this inequality can be made arbitrarily small by taking m sufficiently large since $\frac{1}{2\pi}\int_{-2}^{2}\sqrt{4-x^2}\,dx = 1$, so in fact

$$\lim_{n\to\infty} \int |\xi(x)| \left| E_n(x) - \frac{1}{2\pi}\sqrt{(4-x^2)_+} \right| dx = 0.$$

It follows that

$$\int \xi E_n \to \frac{1}{2\pi}\int_{-2}^{2} \xi(x)\sqrt{(4-x^2)_+}\,dx.$$

We collect our results into the following theorem:

Theorem. $\mathbf{E}\mu_{M_n/\sqrt{n}} = E_n\,dx \to \rho_{\mathrm{sc}}\,dx$ *weakly, where* $\rho_{\mathrm{sc}}(x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}$. *Furthermore,* $E_n \to \rho_{\mathrm{sc}}$ *uniformly on any compact interval contained in* $(-2,2)$ *, and* $E_n \to \rho_{\mathrm{sc}} \equiv 0$ *in* L^1 *on* $\mathbb{R}\setminus[-2,2]$.

7 Connection with the quantum harmonic oscillator

Recall from the preceding section that we denote by P_n the Hermite polynomials orthonormal with respect to $w(x) = e^{-x^2/2}$. Further we have defined $\phi_n = P_n w^{1/2}$ and $\psi_n(x) = n^{1/4}\phi_n(\sqrt{n}x)$, so

$$\int \psi_n^2 = \sqrt{n}\int \phi_n^2(\sqrt{n}x)\,dx = \int \phi_n^2\,dx = \int P_n(x)^2 w(x)\,dx = 1,$$

i.e., $\|\psi_n\|_{L^2} = 1$. In fact the ϕ_n are (modulo horizontal rescaling) the normalized stationary states for the quantum-mechanical harmonic oscillator. See, for example, Griffiths [2] for a discussion of the harmonic oscillator.

In Griffiths [2], it is indicated graphically that the ϕ_n^2 converge in some sense to the density of the classical distribution of the harmonic oscillator. We will formulate and prove this convergence precisely.

Consider the harmonic oscillator defined by

$$x(t) = A\cos(\omega t).$$

It is clear that the amount of time per period for which $x(t) \ge y$ (for $y \in [-A, A]$) is given by $2\omega^{-1}\cos^{-1}(y/A)$, and the period of oscillation is $2\pi/\omega$, so the fraction of time for which $x(t) \ge y$ is given by $1 - \pi\cos^{-1}(y/A)$.

It follows the stationary distribution function F for the position of the harmonic oscillator satisfies $F(y) = 1 - \pi \cos^{-1}(y/A)$ for $y \in [-A, A]$, and the corresponding probability density function is

$$f(y) = F'(y) = \frac{1}{A\pi}\left(1 - \left(\frac{x}{A}\right)^2\right)^{-1/2}.$$

Without loss of much generality, we fix $A = 2$, so

$$f(x) = \frac{1}{2\pi}\left(1 - \frac{x^2}{4}\right)_+^{-1/2}.$$

In fact we must rescale the ϕ_n^2 to obtain convergence to the classical density f. More precisely, we have the following:

Theorem. *For any bounded continuous function $\xi : \mathbb{R} \to \mathbb{R}$, $\int \psi_n^2 \xi \to \int f\xi$.*

Proof. Let $\xi : \mathbb{R} \to \mathbb{R}$ be continuous with $|\xi| \le B$ for some $B > 0$. From the preceding section, we have that on any compact interval I contained in $(-2, 2)$,

$$\psi_n(x)^2 = k(x)^{-1}\left(a(x)e^{in\Psi(x)} + b(x)e^{-in\Psi(x)}\right)^2,$$

where

$$a(x) = a(0) + O\left(n^{-1}\right), \quad b(x) = b(0) + O\left(n^{-1}\right).$$

(Recall that a and b depend on n). Furthermore, we showed that $a(0) \to \frac{1}{2}\pi^{-1/2}$ and $b(0) = (-1)^n a(0)$. Then we have that

$$\psi_n(x)^2 = k(x)^{-1}a(0)^2\left(e^{in\Psi(x)} + (-1)^n e^{-in\Psi(x)} + O\left(n^{-1}\right)\right)^2$$

on I.

Let

$$g_n(x) = k(x)^{-1}a(0)^2\left(e^{in\Psi(x)} + (-1)^n e^{-in\Psi(x)}\right)^2$$

for $x \in (-2, 2)$. We claim that $\int_I \psi_n^2\xi - \int_I g_n\xi \to 0$. Indeed,

$$\psi_n^2(x) - g_n(x) = k(x)^{-1}a(0)^2 \cdot O(n^{-1}),$$

and $a(0)$ is bounded uniformly in n, so

$$\int_I |\psi_n^2 - g_n||\xi| \to 0$$

by bounded convergence, and the claim follows.

Then define

$$h_n(x) = k(x)^{-1}\pi^{-1}\frac{1}{4}\left(e^{in\Psi(x)} + (-1)^n e^{-in\Psi(x)}\right)^2$$

for $x \in (-2,2)$. We claim that $\int_I g_n\xi - \int_I h_n\xi \to 0$. Indeed we have that

$$g_n - h_n = k(x)^{-1}\left(e^{in\Psi(x)} + (-1)^n e^{-in\Psi(x)}\right)^2 \left(a(0)^2 - \frac{1}{4}\pi^{-1}\right),$$

so again our claim follows from bounded convergence.

Now

$$\begin{aligned}
\int_I h_n\xi &= \frac{1}{4\pi}\int_I k(x)^{-1}\left(e^{in\Psi(x)} + (-1)^n e^{-in\Psi(x)}\right)^2 \xi(x)\,dx \\
&= \frac{1}{4\pi}\int_{\Psi(I)} k\left(\Psi^{-1}(x)\right)^{-1}\left(e^{inx} + (-1)^n e^{-inx}\right)^2 \xi\left(\Psi^{-1}(x)\right)\left(\Psi^{-1}\right)'(x)\,dx \\
&= \frac{1}{4\pi}\int \left(e^{inx} + (-1)^n e^{-inx}\right)^2 \zeta(x)\,dx,
\end{aligned}$$

where

$$\zeta(x) = k\left(\Psi^{-1}(x)\right)^{-1}\xi\left(\Psi^{-1}(x)\right)\left(\Psi^{-1}\right)'(x)\chi_{\Psi(I)}(x),$$

so $\zeta \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Then by the Plancherel identity, since

$$\mathcal{F}\left[\left(e^{inx} + (-1)^n e^{-inx}\right)^2\right](u) = (-1)^n\delta\left(u - \frac{2n}{2\pi}\right) + 2\delta(u) + (-1)^n\delta\left(u + \frac{2n}{2\pi}\right),$$

we have that

$$\int_I h_n\xi = \frac{1}{4\pi}\left((-1)^n\left[\hat{\zeta}\left(-\frac{2n}{2\pi}\right) + \hat{\zeta}\left(\frac{2n}{2\pi}\right)\right] + 2\hat{\zeta}(0)\right).$$

By the Riemann-Lebesgue lemma, $(-1)^n\left[\hat{\zeta}\left(-\frac{2n}{2\pi}\right) + \hat{\zeta}\left(\frac{2n}{2\pi}\right)\right] \to 0$ as $n \to \infty$. Furthermore,

$$\begin{aligned}
\hat{\zeta}(0) &= \int_{\Psi(I)} k\left(\Psi^{-1}(x)\right)^{-1}\xi\left(\Psi^{-1}(x)\right)\left(\Psi^{-1}\right)'(x)\,dx \\
&= \int_I k(x)^{-1}\xi(x)\,dx.
\end{aligned}$$

Thus

$$\int_I h_n\xi - \int \frac{1}{2\pi}k(x)^{-1}\xi(x)\,dx \to 0$$

as $n \to \infty$.

We showed in the preceding section that $k(x) \to \left(1 - \frac{x^2}{4}\right)^{1/2}$ uniformly on I as $n \to \infty$, and since k is bounded away from 0 on I it follows that $k(x)^{-1} \to \left(1 - \frac{x^2}{4}\right)^{-1/2} = 2\pi f(x)$ uniformly on I. Therefore

$$\int \frac{1}{2\pi}k(x)^{-1}\xi(x)\,dx - \int f\xi\,dx \to 0$$

as $n \to \infty$.

Putting together our results, we have shown that $\int_I \psi_n^2 \xi \to \int_I f\xi\,dx$ as $n \to \infty$. Now we could have taken $\xi \equiv 1$ in the preceding, so in particular we have that $\int_I \psi_n^2 \to \int_I f$ as $n \to \infty$. Of course, ψ_n^2 is a probability density, so $\int_{\mathbb{R}\setminus I} \psi_n^2 \to 1 - \int_I f$.

Now

$$\begin{aligned}\left|\int_{\mathbb{R}} \psi_n^2 \xi - \int_{\mathbb{R}} f\xi\right| &\leq \left|\int_I \psi_n^2 \xi - \int_I f\xi\right| + \left|\int_{\mathbb{R}\setminus I} \psi_n^2 \xi\right| + \left|\int_{\mathbb{R}\setminus I} f\xi\right| \\ &\leq \left|\int_I \psi_n^2 \xi - \int_I f\xi\right| + B\int_{\mathbb{R}\setminus I} \psi_n^2 + B\int_{\mathbb{R}\setminus I} f \\ &\to 2B\left(1 - \int_I f\right).\end{aligned}$$

Now by taking $I = [-2 + m^{-1}, 2 - m^{-1}]$ for m sufficiently large, we can make the last expression above arbitrarily small (since $\int_I f = 1$). Therefore $\int_{\mathbb{R}} \psi_n^2 \xi \to \int_{\mathbb{R}} f\xi$ as $n \to \infty$, and the proof is complete. □

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