An Introduction of Tutte Polynomial

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Abstract

Tutte polynomial, defined for matroids and graphs, has the important property that any multiplicative graph invariant with a deletion and contraction property is an evaluation of it. In the article we introduce the three definitions of Tutte polynomial, show their equivalence, and present some examples of matroids and graphs to show how to compute their Tutte polynomials. This article is based on [3].
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1 Definitions

Tutte polynomial is named after W. T. Tutte. It is defined on both matroids and graphs. In this paper, we focus on the Tutte polynomial of a matroid. We introduce three equivalent definitions in Section 1, and then show their equivalence in Section 2. Throughout this section, Let $M = (E, r)$ be a matroid, where $r$ is the rank function of $M$.

1.1 Rank-Nullity

**Definition 1.1.** For all $A \subseteq E$, we denote

$$z(A) = r(E) - r(A)$$

and

$$n(A) = |A| - r(A).$$

$n(A)$ is called the nullity of $A$.

For example, if $A$ is a basis of $M$, then

$$z(A) = n(A) = 0.$$ 

Then Tutte polynomial is defined by

**Definition 1.2.**

$$T_M(x, y) = \sum_{A \subseteq E} (x - 1)^{z(A)}(y - 1)^{n(A)}.$$ 

1.2 Deletion-Contraction

We can also define Tutte polynomial in a recursive way. We need the notion of deletion and contraction.

**Definition 1.3.** Let $T$ be a subset of $E$. The deletion of $M$ with respect to $T$, denoted as $M \setminus T$, is a matroid with ground set $E - T$, and independent sets

$$\mathcal{I}(M \setminus T) = \{I \cap (E - T) | I \in \mathcal{I}(M)\}.$$ 

If $T = \{e\}$, we also write $M\setminus e$ for $M\setminus T$.

**Definition 1.4.** Let $T$ be a subset of $E$. The restriction of $M$ with respect to $T$, denoted as $M(T)$, is the matroid

$$M \setminus (E - T).$$

If $T = \{e\}$, we also write $M(e)$ for $M(T)$.

**Definition 1.5.** $M^*$ is a matroid, called the dual of $M$, with ground set $E$, and basis

$$\mathcal{B}(M^*) = \{E - B | B \in \mathcal{B}(M)\}.$$
Definition 1.6. Let $T$ be a subset of $E$. The contraction of $M$ with respect to $T$, denoted as $M/T$, is the matroid 

$$(M^*\setminus T)^*.$$ 

If $T = \{e\}$, we also write $M/e$ for $M/T$.

Definition 1.7. Let $e \in E$, call $e$ a loop of $M$ if $\{e\}$ is not an independent set of $M$; call $e$ a coloop (or isthmus) if every basis of $M$ contains $e$.

It’s trivial to show

Proposition 1.8. Let $e \in E$. If $e$ is either a loop or coloop of $M$, then

$$M/e = M\setminus e.$$  

With these notions, now we can give another definition of Tutte polynomial. We define $T_M(x, y)$ recursively as follows:

Definition 1.9. If $M$ is the empty matroid ($E$ is $\emptyset$), then define

$$T_M(x, y) = 1.$$  

If $e \in E$ is a loop of $M$, then define

$$T_M(x, y) = yT_{M/e}(x, y).$$  

If $e \in E$ is a coloop of $M$, then define

$$T_M(x, y) = xT_{M/e}(x, y).$$  

Finally if $e \in E$ is neither a loop nor a coloop of $M$, then define

$$T_M(x, y) = T_{M/e}(x, y) + T_{M\setminus e}(x, y).$$

This definition is recursive because if $e \in E$ then the ground sets of both $M/e$ and $M\setminus e$ are one element less than $E$.

Definition 1.10. Let $f$ be a function defined on all matroids and $e \in E$. If

$$f(M) = f(M(e))f(M/e)$$ if $e$ is a loop or coloop

and

$$f(M) = f(M\setminus e) + f(M/e)$$ if $e$ is neither a loop nor coloop,

then we call $f$ a Tutte-Grothendieck invariant, T-G invariant for short.

From this deletion-contraction definition we can also see that the Tutte polynomial is a T-G invariant. Actually we can show that any T-G invariant is an evaluation of the Tutte polynomial.[2][6.2.6]
1.3 Internal and External activity

This alternative definition is Tutte’s original definition. It explicitly gives the coefficients of the polynomial. We introduce two relevant notions. In this subsection let’s fix an order $\prec$ on the elements of $E$, say $E = \{e_1, \cdots, e_n\}$, where $e_i < e_j$ if and only if $i < j$.

**Definition 1.11.** Let $B$ be a basis of $M$. An element $e \in B$ is called internally active if $e$ is the smallest element with respect to order $\prec$ in the unique cocircuit disjoint from $B - \{e\}$, otherwise $e$ is called internally passive. Dually an element $f \in E - B$ is called externally active if $f$ is the smallest element in the unique circuit contained in $B \cup \{f\}$, otherwise $f$ is called externally passive. Let $IA(B), IP(B), EA(B), EP(B)$ be the set of internally active, internally passive, externally active and externally passive elements of $B$ respectively.

**Remark 1.12.** We need the following lemma for the existence and uniqueness of the circuit contained in $B \cup \{f\}$.

**Lemma 1.13.** Let $B$ be a basis of $M$ and $f \in E - B$, then there exists a unique circuit $C$ contained in $B \cup \{f\}$ and $f \in C$.

**Proof.** Since $B$ is a basis, so it is a maximal independent set. So $B \cup \{f\}$ is not an independent set. By the definition of circuit there exists a circuit $C$ contained in $B \cup \{f\}$. Of course $C$ is not contained in $B$, so we have $f \in C$. Suppose there is another distinct circuit $C'$ contained in $B \cup \{f\}$, then $f \in C'$ as well. But by the third axiom of circuits we have another circuit $C_2$ such that $C_2 \subseteq C \cup C' - \{f\}$, then $C_2 \subseteq B$, a contradiction. So $C$ is the unique circuit contained in $B \cup \{f\}$.

Now we could carry out our third definition of Tutte polynomials.

**Definition 1.14.** For any nonnegative integers $i, j$ let $t_{i,j}$ be the number of basis of $M$ with exactly $i$ internally active elements and $j$ externally active elements. Then

$$T_M(x, y) = \sum_{i,j \geq 0} t_{i,j} x^i y^j.$$

**Remark 1.15.** The equivalence of these definitions shows that actually $t_{i,j}$ is independent to the choice of order $\prec$.

2 Equivalence of Definitions

In this section we prove the equivalence of the three definitions.

2.1 First and Second Definition

In order to show that first definition is equivalent to the second, it’s enough to show that

$$\sum_{A \subseteq E} (x - 1)^{z(A)} (y - 1)^{n(A)}$$
satisfies the equalities in Definiton 1.9. Actually the equality for empty matroid is trivial. If $E = \{e\}$, then $e$ is either a loop or coloop. If $e$ is a loop, then the sum is

$$(x - 1)^{z(\emptyset)}(y - 1)^{n(\emptyset)} + (x - 1)^{z(\{e\})}(y - 1)^{n(\{e\})} = 1 + (y - 1) = y.$$ 

If $e$ is a coloop, then the sum is

$$(x - 1)^{z(\emptyset)}(y - 1)^{n(\emptyset)} + (x - 1)^{z(\{e\})}(y - 1)^{n(\{e\})} = 1 + (x - 1) = x.$$ 

It’s enough to verify the deletion-contraction relation for an element $e$ that is neither a loop nor a coloop. In this case let $z'$ and $r'$ be the counterparts of $z$ and $r$ for $M/e$. Since $e$ is neither a loop nor a coloop, we have

$$r(E - e) = r(E)$$

and

$$r'(E - e) = r(E) - 1.$$ 

For $A \subseteq E - \{e\}$, we have

$$r'(A) = |A| - r^*(E - e) + r^*(E - e - A)$$

$$= |A| - (|E - e| - r(E) + r(e)) + (|E - e - A| - r(E) + r(A \cup e))$$

$$= |A| - |E| + 1 + r(E) - r(e) + |E| - 1 - |A| - r(E) + r(A \cup e)$$

$$= r(A \cup e) - r(e)$$

$$= r(A \cup e) - 1.$$ 

Then we have

$$T_M(x, y) = \sum_{A \subseteq E} (x - 1)^{z(A)}(y - 1)^{n(A)}$$

$$= \sum_{A \subseteq E - \{e\}} (x - 1)^{z(A)}(y - 1)^{n(A)}$$

$$+ \sum_{A \subseteq E - \{e\}} (x - 1)^{z(A \cup \{e\})}(y - 1)^{n(A \cup \{e\})}$$

$$= \sum_{A \subseteq E - \{e\}} (x - 1)^{r(E - e) - r(A)}(y - 1)^{n(A)}$$

$$+ \sum_{A \subseteq E - \{e\}} (x - 1)^{r(E) - r(A \cup \{e\})}(y - 1)^{|A| + 1 - r(A \cup \{e\})}$$

$$= T_{M/e}(x, y) + \sum_{A \subseteq E - \{e\}} (x - 1)^{r'(E - e) - r'(A)}(y - 1)^{|A| - r'(A)}$$

$$= T_{M/e}(x, y) + T_{M/e}(x, y).$$

So this equivalence is proved.

**Remark 2.1.** The deletion-contraction definition enables us to calculate the Tutte polynomial of a matroid, though this matroid is not efficient when the ground set is very large. We will discuss this later in section 3.
2.2 First and Third Definition

In this subsection we show the following equivalence:

**Theorem 2.2.** Fix an order $\prec$ on the ground set $E$, we have

$$
\sum_{i,j \geq 0} t_{i,j} x^i y^j = \sum_{A \subseteq E} (x - 1)^{z(A)} (y - 1)^{n(A)}.
$$

We present the proof given in [1]. We need to introduce the notion of *shelling*. The definitions of simplicial complex, facet, and pure are from [1] as well.

**Definition 2.3.** Let $\Delta$ be a pure simplicial complex. A *shelling* of $\Delta$ is a linear sequence $F_1, F_2, \cdots, F_t$ of the facets of $\Delta$ such that for all $1 \leq i < j \leq t$ there exists a facet $F_k$ satisfying $1 \leq k < j$ and an element $x \in F_j$ such that

$$
F_i \cap F_j \subseteq F_k \cap F_j = F_j - \{x\}.
$$

A complex is called *shellable* if it is pure and admits a shelling. If $F_1, F_2, \cdots, F_t$ is a shelling of $\Delta$, for $1 \leq i \leq t$ we define

$$
\Delta_i = \{G \in \Delta | G \subseteq F_k \text{ for some } k \leq i\}
$$

and

$$
\mathcal{R}(F_i) = \{x \in F_i | F_i - \{x\} \in \Delta_{i-1}\}.
$$

$\mathcal{R}(F_i)$ is called the *restriction* of $F_i$.

Then we have the following lemma.

**Lemma 2.4.** The intervals $[\mathcal{R}(F_i), F_i]$ partition the shellable complex $\Delta$, where interval $[G_1, G_2]$ is defined as $\{G \in \Delta | G_1 \subseteq G \subseteq G_2\}$.

**Proof of lemma.** It’s enough to show that

$$
[\mathcal{R}(F_i), F_i] = \Delta - \Delta_{i-1}.
$$

Let $G \in [\mathcal{R}(F_i), F_i]$. Then $G \subseteq F_i$, so $G \in \Delta_i = \Delta$. Suppose $G \subseteq F_i$ for some $i \leq t - 1$. Then by definition there exists $i < j \leq t - 1$ and an element $x \in F_i$ such that

$$
G \subseteq F_i \cap F_i \subseteq F_j \cap F_i = F_j - \{x\}.
$$

But then $F_i - \{x\} \subseteq F_j$, so $F_i - \{x\} \in \Delta_{t-1}$, so $x \in \mathcal{R}(F_i) \subseteq G$, a contradiction. Hence $G \notin \Delta_{i-1}$ which means

$$
G \in \Delta - \Delta_{i-1}.
$$

On the other hand let $H \in \Delta - \Delta_{i-1}$, then $H \subseteq F_i$. For any $x \in \mathcal{R}(F_i)$, there exists some $j \leq t - 1$ such that $F_i - \{x\} \subseteq F_j$. Since $H \notin \Delta_{i-1}$, we have $H \notin F_j$, so $x \in H$. Therefore $\mathcal{R}(F_i) \subseteq H$, we have

$$
H \in [\mathcal{R}(F_i), F_i].
$$
By the definition of matroid, it’s obvious that if we regard all independent sets as faces, then we have a simplicial complex, called the matroid complex. The facets are just the basis of matroid. Since all basis of a matroid have the same cardinality, matroid complexes are pure. In addition if given a order $\prec$ on the ground set, then it induces an order $\prec$ on the set of basis of $M$, that is, if \{$x_1 < x_2 < \cdots < x_r$\} and \{$y_1 < y_2 < \cdots < y_r$\} are two distinct basis, then $x < y$ if there is some $e$ such that

$$x_i = y_i \text{ for all } 1 \leq i \leq e - 1 \text{ and } x_e < y_e.$$ 

We claim that the sequence of basis in lexicographic order is a shelling of the matroid complex. It follows from the lemma below.

**Lemma 2.5.** If $B$ and $C$ are two basis of $M$ such that $B \prec C$, then there exists another basis $A \prec C$ such that $B \cap C \subseteq A \cap C$ and $|A \cap C| = |C| - 1$.

**Proof.** Suppose $B = \{b_1 < b_2 < \cdots < b_e\}$ and $C = \{c_1 < c_2 < \cdots < c_r\}$, where $b_i = c_i$ for all $1 \leq i \leq e - 1$ and $b_e < c_e$. Then $b_e < c_i$ for all $e \leq i \leq r$. So $b_e \notin C$. By the second axiom of basis, there exists an element $a \in C - B$ such that $(C - \{a\}) \cup \{b_e\}$ is also a basis of $M$. We just take $A$ to be this basis. \qed

So we have

**Proposition 2.6.** Given an ordering $\prec$ on the ground set of a matroid, then the corresponding matroid complex is shellable.

Now we note that for any basis $B$

$$\mathcal{R}(B) = IP(B).$$

**Lemma 2.7.** Given an ordering $\prec$ on the ground set of a matroid, then the family of intervals $[IP(B), E - EP(B)]$ partitions the Boolean algebra of subsets of $E$.

**Proof.** Let $A \subseteq E$, then it’s enough to show that there exists a unique basis $B_A$ such that

$$IP(B_A) \subseteq A \subseteq E - EP(B_A).$$

Let $X_A$ be the lexicographically greatest basis in the submatroid $M|_A$ that is induced on $A$ by $M$. Then subset $X_A$ is independent in $M$, so it’s a face in the matroid complex of $M$. By Lemma 2.4 and Proposition 2.6, there exists a unique basis $B_A$ such that

$$IP(B_A) = \mathcal{R}(B_A) \subseteq X_A \subseteq B_A. \tag{1}$$

Next we show that

$$A \subseteq E - EP(B_A).$$

Suppose $a \in A \cap EP(B_A)$, then $a \in A - B_A = A - X_A$ (this equality is from the maximality of $X_A$). Since $a$ is externally passive in $B_A$, there exists another element $b \in B_A$ such that $b < a$ and $(B_A - b) \cup a$ is also basis of $M$. If $b \notin A,$
then $X_A \cup \{a\}$ is contained in a basis of $M$, so it would be independent in $M$, which is a contradiction to the maximality of $X_A$; If $b \in A$, then

$$(X_A - b) \cup a$$

is contained in a basis of $M$, so it would be independent in $M$. Then we find a basis of $M|_A$ that is lexicographically greater than $X_A$, a contradiction too. Hence $A \cap EP(B_A) = \emptyset$, we have showed that

$$IP(B_A) \subseteq A \subseteq E - EP(B_A).$$

Finally we show the uniqueness. Suppose there is another basis $B$ such that

$$IP(B) \subseteq A \subseteq E - EP(B),$$

we claim that $X = B \cap A$ is a basis in $M|_A$. It’s enough to show for any element $a \in A - B$,

$$(B \cup \{a\}) \cap A$$

is a dependent set in $M$. Since $A \cap EP(B) = \emptyset$, we know that $a \in EA(B)$, which means $a$ in the greatest element in the unique circuit $C \subseteq B \cup \{a\}$. Then for any $c \in C$, we have $a < c$ and $(B - \{c\}) \cup \{a\}$ is also a basis of $M$. By definition $c \in IP(B) \subseteq A$. Hence $C \subseteq A, C \subseteq (B \cup \{a\}) \cap A$, our claim is proved.

Next we show that

$$X = X_A.$$  

Suppose not, since $X_A$ is the lexicographically greatest basis in $M|_A$, then

$$X_A < X.$$  

So we may assume

$$X = \{x_1, x_2, \cdots, x_k\}$$ and $$X_A = \{y_1, y_2, \cdots, y_k\},$$

where these is some $e$ such that $x_i = y_i$ for all $1 \leq i \leq e - 1$ and $x_e < y_e$. Then $x_e < y_j$ for all $j \geq e$. By the basis exchange axiom there exists some $j \geq e$ such that

$$(X - \{x_e\}) \cup \{y_j\}$$

is also a basis of $M|_A$. Then it is contained in some basis of $M$, which means $y_j$ is externally passive in $B$, and $y_j \in X \subseteq A$, so $y_j \in A \cap EP(B)$, a contradiction. Therefore we have $X = X_A$. Then from $IP(B) \subseteq A \cap B = X$ and $X = A \cap B \subseteq B$ we have

$$IP(B) \subseteq X_A \subseteq B.$$  

By (1) we deduce that $B = B_A$. \qed

Proof of Theorem 2.2. Let $B$ be a basis and $IP(B) \subseteq A \subseteq E - EP(B)$. By previous proof we have the following result

**Proposition 2.8.** $B \cap A$ is a basis in $M|_A$.  


Therefore \( r(A) = |A \cap B| \).

So \( z(A) = |B - A| \).

Dually \( n(A) = z^*(E - A) = |E - B| - |E - A| = |A - B| \).

Hence
\[
\sum_{A \subseteq E} (x - 1)^{z(A)} (y - 1)^{n(A)}
= \sum_{B \text{ is basis}} \sum_{\emptyset \subseteq B \subseteq E} \left( x - 1 \right)^{|B - A|} \left( y - 1 \right)^{|A - B|}
= \sum_{B \text{ is basis}} \sum_{0 \leq i \leq |A(B)|, 0 \leq j \leq |E(A)|} \binom{|A(B)|}{i} \binom{|E(A)|}{j} (x - 1)^i (y - 1)^j
= \sum_{B \text{ is basis}} x^{|I A(B)|} y^{|E A(B)|}
= \sum_{i,j \geq 0} t_{i,j} x^i y^j.
\]

Corollary 2.9. The coefficients \( t_{i,j} \) of Tutte polynomial are independent of the choice of ordering \( \prec \).

3 Examples

In this section we present some examples of matroids to show their Tutte polynomials.

3.1 Dual Matroids

First we analyze the relation between \( T_M(x, y) \) and \( T_{M^*}(x, y) \), where \( M^* \) is the dual of \( M \). Let \( z^* \) and \( n^* \) be the counterparts of functions \( z \) and \( n \) in \( M^* \). Then for any \( A \subseteq E \) we have
\[
z^*(A) = r^*(E) - r^*(A)
= (|E| - r(E)) - (r(E - A) + |A| - r(E))
= |E| - r(E - A) - |A|
= |E - A| - r(E - A)
= n(E - A).
\]

where we use the following proposition from [4][Prop. 2.1.9]:
**Proposition 3.1.** For all $A \subseteq E$,

$$r^*(A) = r(E - A) + |A| - r(E).$$

Dually we have

$$n^*(A) = z(E - A).$$

Therefore

$$T_{M^*}(x, y) = \sum_{A \subseteq E} (x - 1)^{r^*(A)}(y - 1)^{n^*(A)}$$

$$= \sum_{A \subseteq E} (x - 1)^{n(E - A)}(y - 1)^{z(E - A)}$$

$$= \sum_{A \subseteq E} (x - 1)^{n(A)}(y - 1)^{z(A)}$$

$$= T_M(y, x).$$

So we have the following statement:

**Proposition 3.2.**

$$T_{M^*}(x, y) = T_M(y, x).$$

### 3.2 Uniform Matroids

Uniform matroid is a kind of matroids with very simple structure. Let $0 \leq m \leq n$, where $m, n$ are integers. The uniform matroid $U_{m,n}$ is a matroid with an $n$-element ground set $E$ and all basis are exactly the family of $m$-element subsets of $E$. From the rank-nullity definition we can easily obtain the following formula of the Tutte polynomial of $U_{m,n}$.

$$T_{U_{m,n}}(x, y) = \sum_{i=0}^{m-1} \binom{n}{i} (x - 1)^{m-i} + \sum_{i=m+1}^{n} \binom{n}{i} (y - 1)^{i-m}.$$ 

While from the internal and external activity definition we can compute the value of $t_{i,j}$. Given a basis $B$ of $M$. Suppose

$$B = \{b_1 < b_2 < \cdots < b_m\}$$

and

$$E - B = \{c_1 < c_2 < \cdots < c_{n-m}\}$$

Then by definition, $b_i$ is internal active iff $b_i$ is the smallest element among $b_i \cup (E - B)$, which means

$$b_i < c_1.$$ 

Similarly $c_j$ is external active iff

$$c_j < b_1.$$
Since either $b_1$ or $c_1$ is 1, so

$$t_{i,j} = 0 \text{ if } ij \neq 0.$$  

In addition if $b_1 = 1$, then $B$ has $c_1 - 1$ internal active elements. So

$$t_{i,0} = \left(\frac{n - i - 1}{m - i}\right) \text{ for all } i > 0.$$  

and

$$t_{0,j} = \left(\frac{n - j - 1}{m - 1}\right) \text{ for all } j > 0.$$  

Finally it’s obvious that

$$t_{0,0} = 0.$$  

So we have

$$T_{U_{n,m}}(x, y) = \sum_{i=0}^{m} \left(\frac{n - i - 1}{m - i}\right)x^i + \sum_{j=0}^{m} \left(\frac{n - j - 1}{m - 1}\right)y^j.$$  

### 3.3 An example of graphic matroids

If $G$ is a graph, then $M[G]$ is called a graphic matroid whose ground set is $E(G)$ and the circuits are the set of minimal cycles in $G$. For graphic matroids the deletion-contraction definition of Tutte polynomial is very useful because we have the following result.

**Proposition 3.3.** Let $e \in E(G)$, $u$ and $v$ are the two (possibly identical) vertices connected by $e$. Let $G/e$ be the graph obtained from $G$ by deleting $e$ and identifying the two vertices $u, v$. Then we have

$$M[G/e] = M[G]/e.$$  

**Proof.** It’s enough to show $\mathcal{I}(M[G/e]) = \mathcal{I}(M[G]/e)$. For any $A \subseteq E - \{e\}$, $A$ is an independent set in $M[G]/e$ iff $A \cup \{e\}$ contains no cycle in $G$ ([4][Prop. 3.1.7]). While $A \cup \{e\}$ contains no cycle in $G$ iff $A$ contains no cycle in $G/e$.  

Now by proposition 3.3 and deletion-contraction definition, we have the formula for Tutte polynomials of graphic matroids.

**Proposition 3.4.** Let $G$ be a graph and $e$ is an edge of $G$. If $e$ is neither a loop nor a coloop in $M[G]$, then

$$T_{M[G]}(x, y) = T_{M[G-e]}(x, y) + T_{M[G/e]}(x, y),$$

where $G - e$ is the graph obtained from $G$ by deleting the edge $e$.  

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Next we present an example of graphic matroids and compute their Tutte polynomials. Let $T_n$ be the simple graph with $n$ non-intersecting triangles having a common vertex:

We use proposition 3.4 to compute $T_M[T_n](x,y)$. Pick an arbitrary edge $e$ in $T_n$, then we need to look at graphs $T_n - e$ and $T_n/e$. Notice that $T_n - e$ is obtained from $T_{n-1}$ by adding two coloops, so

$$T_M[T_n - e](x,y) = x^2 T_M[T_{n-1}](x,y).$$

As for $T_n/e$, we can pick another edge in the two-edge cycle, say $f$, and then look at $T_n/e - f$ and $T_n/e/f$.

$T_n/e - f$ is obtained from $T_{n-1}$ by adding one coloop, so

$$T_M[T_n/e - f](x,y) = x T_M[T_{n-1}](x,y).$$

$T_n/e/f$ is obtained from $T_{n-1}$ by adding one loop, so

$$T_M[T_n/e/f](x,y) = y T_M[T_{n-1}](x,y).$$

Therefore we have

$$T_M[T_n](x,y) = T_M[T_n - e](x,y)+T_M[T_n/e - f](x,y)+T_M[T_n/e/f](x,y) = (x^2 + x + y) T_M[T_{n-1}](x,y).$$

Figure 1: $T_n$

Figure 2: $T_n - e$
Finally note that $M[T_0]$ is the empty matroid, so

$$T_M[T_n](x, y) = (x^2 + x + y)^n T_M[T_0](x, y) = (x^2 + x + y)^n.$$  

References


