Introduction to Ryser's Conjecture

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Hypergraph

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Definition

A hypergraph $\mathcal{H}$ is a pair (V, E), where V is a set of *vertices* and E is a family of subsets of V. Those subsets in E are called *edges* of $\mathcal{H}$.

Examples of Hypergraphs

Example ($\mathcal{H}_1$)

$V = \{v_1, v_2, v_3, v_4, v_5\}$
$E = \{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_2, v_3, v_4, v_5\}\}.$

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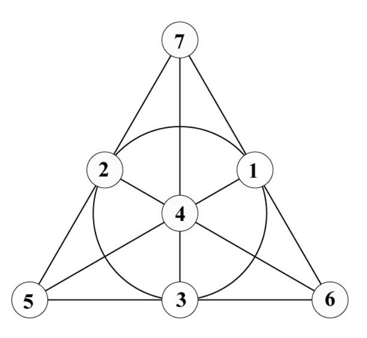
Example ($\mathcal{H}_2$,Fano Plane)



7 vertices, 7 edges.

Uniform and Partite Hypergraphs

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A hypergraph $\mathcal{H} = (V, E)$ is *r-partite* if V admits a partition $V = \bigcup_{i=1}^{r} V_i$ such that for $e \in E$ and $1 \leq i \leq r$, $|e \cap V_i| = 1$.

Examples of Hypergraphs

Example ($\mathcal{H}_3 = K_{n,n,n}$)

The complete 3-partite hypergraph $K_{n,n,n}$.
$V = \{A_1, \cdots, A_n, B_1, \cdots, B_n, C_1, \cdots, C_n\}$,
$E = \{\{A_i, B_j, C_k\} | 1 \leq i, j, k \leq n\}$.

Matching Number

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Examples:

- $\nu(\mathcal{H}_2) = 1$.
- $\nu(\mathcal{H}_3) = n$.

Covering Number

Definition

A *cover* (or *transversal*) of hypergraph $\mathcal{H}$ is a subset W of V such that all edges intersect W. The *covering number* $\tau(\mathcal{H})$ is the minimal cardinality of a covering of $\mathcal{H}$.

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Examples:

- $\tau(\mathcal{H}_2) = 3$.
- $\tau(\mathcal{H}_3) = n$.

Trivial Estimate

Proposition

$\mathcal{H}$ is an r-partite hypergraph. Then

$$\nu(\mathcal{H}) \leq \tau(\mathcal{H}) \leq r\nu(\mathcal{H}).$$

Main Conjecture

Conjecture (Ryser)

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Remark

This conjecture first appeared in J.R. Henderson's Ph.D. thesis(1971), whose advisor is Herbert John Ryser. At around the same time Lovász independently conjectured a stronger statement.

Equality Can Hold

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Proof.

(Sketch) Consider a projective plane with order $r-1$. Delete one vertex and all edges containing it, the remaining vertices of these edges form a partition. And all remaining edges form an r-partite hypergraph $\mathcal{H}$ with $\nu(\mathcal{H}) = 1$ and $\tau(\mathcal{H}) = r-1$. □

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Progress on Main Conjecture

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Ryser's conjecture remains open for $r \geq 4$.

Progress on Main Conjecture

Theorem (Haxell & Scott, 2012)

For $r = 4$ and $r = 5$ if $\mathcal{H}$ is an r-partite hypergraph. Then there exists $\epsilon > 0$ such that

$$\tau(\mathcal{H}) \leq (r - \epsilon)\nu(\mathcal{H}).$$

Intersecting Case

We could consider one particular case of the main conjecture, when

$$\nu(\mathcal{H}) = 1.$$

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Conjecture (Intersecting Case)

$\mathcal{H}$ is an r-partite intersecting hypergraph. Then

$$\tau(\mathcal{H}) \leq r - 1.$$

Progress on Intersecting Case

Theorem (Tuza, 1978, unpublished)

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The intersecting case remains open for $r \geq 6$.

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Mansour, Song and Yuster introduced the following definition:

Definition

For integers $r \geq 2$, let $f(r)$ be the smallest number of edges of any r-partite intersecting hypergraph with $\tau(\mathcal{H}) \geq r - 1$.

Examples of $f(r)$ When r is Small

Proposition

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$$f(4) = 6, f(5) = 9, 12 \leq f(6) \leq 15.$$

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Theorem (Lin, 2013, unpublished)

$$f(6) = 13.$$

A 6-partite intersecting hypergraph $\mathcal{H}$ with 13 edges and $\tau(\mathcal{H}) = 5$

Let $V_1, V_2, V_3, V_4, V_5, V_6$ be the six subsets of V, then each of them gives a partition of the edges, based on which vertex of the subset is contained by one edge. Each small subset corresponds to one vertex.

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V_1	{1,2,3,4}	{7,9,J,Q}	{8,K}	{6,10}	{5}
V_2	{1,5,6,7}	{10,3,J,K}	{2,Q}	{9,4}	{8}
V_3	{1,8,9,10}	{4,6,Q,K}	{5,J}	{3,7}	{2}
V_4	{2,8,6,J}	{4,10,5,Q}	{1,K}	{3,9}	{7}
V_5	{5,8,3,Q}	{7,10,2,K}	{1,J}	{6,9}	{4}
V_6	{2,5,9,K}	{4,7,8,J}	{1,Q}	{3,6}	{10}

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V_3	{1,8,9,10}	{4,6,Q,K}	{5,J}	{3,7}	{2}
V_4	{2,8,6,J}	{4,10,5,Q}	{1,K}	{3,9}	{7}
V_5	{5,8,3,Q}	{7,10,2,K}	{1,J}	{6,9}	{4}
V_6	{2,5,9,K}	{4,7,8,J}	{1,Q}	{3,6}	{10}

To verify this example,

- (intersecting) Every pair of edges is contained by at least one small subset.
- ($\nu(\mathcal{H}) > 4$) The union of any 4 small subsets is not E.

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Theorem (Mansour et al., 2009)

$$f(r) \geq (3 - \frac{1}{\sqrt{18}})r(1 - o(1)) \approx 2.764r(1 - o(1)).$$

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Conjecture (Mansour et al., 2009)

$$f(r) = \Theta(r).$$

Thank you!