Almost-Toric Hypersurfaces

Bo Lin

University of California, Berkeley

September 29th, 2014
Computational Algebraic Geometry Seminar
An almost-toric hypersurface is parameterized by monomials multiplied by polynomials in one extra variable. We determine the Newton polytope of such a hypersurface, and apply this to give an algorithm for computing the implicit equation.
Let $A$ be a $n \times (n + 2)$ full-rank matrix with integer entries.

\[ A = \begin{bmatrix} a_0 & a_1 & \ldots & a_{n+1} \end{bmatrix}. \]
Toric Part of $Z_{A,f}$

Let $A$ be a $n \times (n+2)$ full-rank matrix with integer entries.

$$A = \begin{bmatrix} a_0 & a_1 & \ldots & a_{n+1} \end{bmatrix}.$$ 

If the entries of each vector $a_i$ have the same sum $d$, then

$$t^{a_0}, t^{a_1}, \ldots, t^{a_{n+1}}$$

are Laurent monomials in $n$ variables of the same degree $d$

$$t_1, \ldots, t_n,$$

where

$$t^{a_i} = t_1^{a_{1,i}} t_2^{a_{2,i}} \ldots t_n^{a_{n,i}}.$$
We fix $K$ as an algebraically closed field. Let

$$f_0, f_1, \ldots, f_{n+1} \in K[x]$$

be univariate polynomials in another variable $x$. 
An $n$-dimensional almost-toric hypersurface $Z_{A,f}$ is a codimension 1 hypersurface that is the Zariski closure of following parameterization:

$$Z_{A,f} = \text{cl}(\{(t^0 f_0(x) : \ldots : t^{n+1} f_{n+1}(x)) \in \mathbb{P}^{n+1} | t \in (K^*)^n, x \in K\}),$$

Remark

We need some mild hypothesis of $A, f$ to guarantee that $Z_{A,f}$ is a codimension 1 hypersurface. We will see the hypothesis in our main theorem.
Suppose $Z_{A,f}$ is an almost-toric hypersurface, then its ideal is principal. The Newton polytope of $Z_{A,f}$ is defined as the Newton polytope of the generator (up to a scalar multiple) of this principal ideal. This generator, denoted as $p(u_0, \ldots, u_{n+1})$, is called the *implicit polynomial* of $Z_{A,f}$.

**Proposition**

*The Newton polytope of an almost-toric hypersurface $Z_{A,f}$ is a polygon in $\mathbb{R}^{n+2}$. Its vertices are non-negative lattice points.*
Proof.

If we substitute $u_i = t^{a_i} f_i(x)$ in $p$, we get a polynomial $q$ in variables $t_1, \ldots, t_n, x$. Actually $q$ must be the zero polynomial. Since $p$ is the generator of a principal ideal, each monomial in $q$ has the same degree of $t_1, t_2, \ldots, t_n$, then each edge of the Newton polytope is in the null space of $A$, which is a 2-dimensional linear subspace. So the Newton polytope is a (possibly degenerate) polygon.
The Plücker matrix $P_A$ of $A$ is a $(n+2) \times (n+2)$ square matrix whose row vectors span the kernel of $A$. The $i, j$-th entry of $P_A$ is

$$p_{i,j} = \begin{cases} 
\frac{1}{\delta}(-1)^{i+j} \det(A[i,j]), & i < j; \\
-p_{j,i}, & i > j; \\
0, & i = j.
\end{cases}$$

where $A[i,j]$ is the submatrix obtained from $A$ by deleting the $i$-th and $j$-th columns of $A$ and $\delta$ is the greatest common divisor of all these $\det(A[i,j])$. 
Example:

\[ A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}. \]

Then \( \delta = 3 \) and

\[ P_A = \begin{bmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -3 & 2 \\ -2 & 3 & 0 & -1 \\ 1 & -2 & 1 & 0 \end{bmatrix}. \]
Example:

\[ A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}. \]

Then \( \delta = 3 \) and

\[ P_A = \begin{bmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -3 & 2 \\ -2 & 3 & 0 & -1 \\ 1 & -2 & 1 & 0 \end{bmatrix}. \]

**Proposition**

\( P_A \) is skew-symmetric. The rank of \( P_A \) is 2. The entries in each row and column of \( P_A \) sum to 0.
The valuation matrix of $Z_{A,f}$ is defined from the polynomials $f_0, f_1, \ldots, f_{n+1}$. Suppose $g_1, g_2, \ldots, g_m$ are all irreducible factors of $\prod_{i=0}^{n+1} f_i$. Then we define vectors

$$u_j = \deg(g_j) \cdot (\text{ord}_{g_j} f_0, \text{ord}_{g_j} f_1, \ldots, \text{ord}_{g_j} f_{n+1}) \in \mathbb{N}^{n+2}$$

for $1 \leq j \leq m$.

Now we need to simplify the set of these vectors. We combine the pairwise linearly dependent vectors, because they correspond to the same edge of the polygon.
Now let $S = \{u_1, u_2, \ldots, u_m\}$. If two vectors in $S$ are linearly dependent, then we delete them and add their sum to the set. We repeat this procedure. After finite steps, we end up with another set without pairwise linearly dependent vectors. Finally we get 

$$S' = \{v_1, v_2, \ldots, v_l\}.$$ 

**Definition**

*The valuation matrix of $Z_{A,f}$ is*

$$V_f = \begin{bmatrix} v_1^T & v_2^T & \ldots & v_l^T & (-\sum_{j=1}^{l} v_j)^T \end{bmatrix}.$$ 

The last vector represents the valuation at $\infty$. 
Now let $S = \{u_1, u_2, \ldots, u_m\}$. If two vectors in $S$ are linearly dependent, then we delete them and add their sum to the set. We repeat this procedure. After finite steps, we end up with another set without pairwise linearly dependent vectors. Finally we get 

$$S' = \{v_1, v_2, \ldots, v_l\}.$$ 

**Definition**

The valuation matrix of $Z_{A,f}$ is

$$V_f = \begin{bmatrix} v_1^T & v_2^T & \ldots & v_l^T & (-\sum_{j=1}^l v_j)^T \end{bmatrix}.$$ 

The last vector represents the valuation at $\infty$.

**Proposition**

The sum of each row in $V_f$ is zero.
Valuation Matrices

Example:

\[ n = 2, f_0 = x - 1, f_1 = (x - 1)^2(x + 1), f_2 = (x + 1)x^3, f_3 = (x - 2)x. \]

Then the irreducible factors are \( x - 1, x, x + 1, x - 2. \)

1. \( x - 1 \) corresponds to \( (1, 2, 0, 0) \)
2. \( x \) corresponds to \( (0, 0, 3, 1) \)
3. \( x + 1 \) corresponds to \( (0, 1, 1, 0) \)
4. \( x - 2 \) corresponds to \( (0, 0, 0, 1) \)

These vectors are pairwisely linearly independent, so the valuation matrix is

\[
V_f = \begin{bmatrix}
1 & 0 & 0 & 0 & -1 \\
2 & 0 & 1 & 0 & -3 \\
0 & 3 & 1 & 0 & -4 \\
0 & 1 & 0 & 1 & -2
\end{bmatrix}.
\]
Theorem

Suppose $Z_{A,f}$ is defined as before. It has a Plücker matrix $P_A$ for its toric part and a valuation matrix $V_f$ for its coefficients.

(a) if $\text{rank}(P_A \cdot V_f) = 0$ then $Z_{A,f}$ is not a hypersurface;

(b) if $\text{rank}(P_A \cdot V_f) = 1$ then $Z_{A,f}$ is a toric hypersurface;

(c) if $\text{rank}(P_A \cdot V_f) = 2$ then $Z_{A,f}$ is a hypersurface but not toric. The directed edges of the Newton polygon of $Z_{A,f}$ are the nonzero column vectors of $P_A \cdot V_f$. 

Remark

The sum of each row in $P_A \cdot V_f$ is zero.
Main Theorem

Theorem

Suppose $Z_{A,f}$ is defined as before. It has a Plücker matrix $P_A$ for its toric part and a valuation matrix $V_f$ for its coefficients.

(a) if $\text{rank}(P_A \cdot V_f) = 0$ then $Z_{A,f}$ is not a hypersurface;
(b) if $\text{rank}(P_A \cdot V_f) = 1$ then $Z_{A,f}$ is a toric hypersurface;
(c) if $\text{rank}(P_A \cdot V_f) = 2$ then $Z_{A,f}$ is a hypersurface but not toric. The directed edges of the Newton polygon of $Z_{A,f}$ are the nonzero column vectors of $P_A \cdot V_f$.

Remark

The sum of each row in $P_A \cdot V_f$ is zero.
3-dimensional hypersurface $Z_{A,f}$

Let $Z_{A,f}$ admit the following parameterization over $\mathbb{C}$:

$$(t_1^2(x^2+1) : t_1t_2x^3(x-1) : t_1t_3x(x+1) : t_2^2(x-2)(x^2+1) : t_3^2(x-1)^2(x+1)).$$
In this example

\[
A = \begin{bmatrix}
2 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 2 \\
\end{bmatrix}, \quad d = \delta = 2.
\]

Then

\[
P_A = \begin{bmatrix}
0 & -2 & 2 & 1 & -1 \\
2 & 0 & -4 & 0 & 2 \\
-2 & 4 & 0 & -2 & 0 \\
-1 & 0 & 2 & 0 & -1 \\
1 & -2 & 0 & 1 & 0 \\
\end{bmatrix}.
\]
Valuation Matrix of $Z_{A,f}$

In this example the irreducible factors are 

$$x, x - 1, x + i, x - i, x + 1, x - 2.$$ 

But we can combine $x \pm i$ into $x^2 + 1$. So the vectors are 

$$(0, 3, 1, 0, 0), (0, 1, 0, 0, 2), (2, 0, 0, 2, 0),$$

$$(0, 0, 1, 0, 1), (0, 0, 0, 1, 0), (-2, -4, -2, -3, -3).$$

The valuation matrix of $Z_{A,f}$ is 

$$V_f = \begin{bmatrix}
0 & 0 & 2 & 0 & 0 & -2 \\
3 & 1 & 0 & 0 & 0 & -4 \\
1 & 0 & 0 & 1 & 0 & -2 \\
0 & 0 & 2 & 0 & 1 & -3 \\
0 & 2 & 0 & 1 & 0 & -3
\end{bmatrix}.$$
The main theorem tells us that these directed edges are the column vectors of $P_A \cdot V_f$.

The product is

$$
\begin{bmatrix}
-4 & -4 & 2 & 1 & 1 & 4 \\
-4 & 4 & 4 & -2 & 0 & -2 \\
12 & 4 & -8 & 0 & -2 & -6 \\
2 & -2 & -2 & 1 & 0 & 1 \\
-6 & -2 & 4 & 0 & 1 & 3
\end{bmatrix}
$$
Implicit Polynomial of $Z_{A,f}$

Using ideal elimination in Macaulay2 we can compute the implicit polynomial of $Z_{A,f}$ in variables $u_0, u_1, u_2, u_3, u_4$:

\[
16u_1^4u_2^4u_3^2 - 40u_0u_1^4u_2^4u_3u_4 + 8u_0^2u_1^2u_2^4u_3^2u_4 - 16u_0u_1u_2^2u_3u_4^2 \\
+ 20u_0^2u_1^4u_2^2u_3^2u_4^2 + 159u_0^3u_1^2u_2^2u_3^2u_4^2 \quad + u_0^4u_2^2u_3^2u_4^2 \\
+ 54u_0^2u_1^6u_2u_3^3u_4 - 77u_0^3u_1^2u_2u_3^2u_4^3 \quad + 379u_0^4u_1^2u_2u_3^2u_4^3 \\
+ 5u_0^2u_1^8u_2u_4^4 - 27u_0^3u_1^2u_2^2u_3u_4^4 \quad - 29u_0^4u_1^4u_2u_3^4u_4^4 \\
+ 163u_0^5u_1^2u_2^2u_3^3u_4^4 \quad - 12u_0^3u_1^4u_2^2u_4^5 \quad - 35u_0^4u_1^6u_2u_3^4u_4^5 \\
- 425u_0^5u_1^4u_2^2u_3^4u_4^5 \quad + 4u_0^6u_1^2u_2^6u_3^3u_4^5 \quad + 87u_0^5u_1^6u_2^2u_3^6u_4 \\
+ 717u_0^6u_1^4u_2^2u_3^6u_4^6 \quad + 103u_0^6u_1^6u_2^2u_3^7u_4^6 \quad - 115u_0^7u_1^4u_2^2u_3^7u_4^7 \\
+ 12u_0^7u_1^6u_3u_4^8 \quad + 4u_0^8u_1^4u_3^2u_4^8.
\]
The vertices of Newton polytope of this defining polynomial are

\((0, 4, 16, 2, 0), (2, 8, 8, 0, 4), (3, 8, 0, 6, 5), (7, 6, 0, 1, 8), (8, 4, 0, 2, 8), (4, 0, 12, 4, 2)\).

Then the directed edges are

\((2, 4, -8, -2, 4), (-4, 4, 4, -2, -2), (-4, -4, 12, 2, -6), (1, -2, 0, 1, 0), (4, -2, -6, 1, 3), (1, 0, -2, 0, 1)\).
Let $K$ be a field with valuation and

$$f(x_1, \ldots, x_n) = \sum_{u \in \mathbb{N}^n} c_u \prod_{i=1}^{n} x_i^{u_i} \in K[x_1, \ldots, x_n]$$

be a polynomial.
Tropical variety

Let $K$ be a field with valuation and

$$f(x_1, \ldots, x_n) = \sum_{u \in \mathbb{N}^n} c_u \prod_{i=1}^{n} x_i^{u_i} \in K[x_1, \ldots, x_n]$$

be a polynomial. Then for $w \in \mathbb{R}^n$, the tropical polynomial is defined as

$$\text{trop}(f)(w) = \min_{u \in \mathbb{N}^n} (u_i w_i + \text{val}(c_u)).$$
Tropical variety

Let \( K \) be a field with valuation and

\[
f(x_1, \ldots, x_n) = \sum_{u \in \mathbb{N}^n} c_u \prod_{i=1}^{n} x_i^{u_i} \in K[x_1, \ldots, x_n]
\]

be a polynomial. Then for \( w \in \mathbb{R}^n \), the tropical polynomial is defined as

\[
trop(f)(w) = \min_{u \in \mathbb{N}^n} (u_i w_i + \text{val}(c_u)).
\]

The tropical hypersurface is defined as

\[
trop(V(f)) = \{ w \in \mathbb{R}^n | \text{minimum is attained at least twice in trop}(f)(w) \}.
\]

If \( V \) is a variety, the tropical variety \( trop(V) \) is defined as

\[
trop(V) = \bigcap_{f \in I(V)} trop(V(f)).
\]
\[ X_A \text{ and } Y_f \]

**Definition**

\( X_A \) is defined as the Zariski closure of the following parameterization:

\[ (t^{a_0} : t^{a_1} : \ldots : t^{a_{n+1}}). \]

Then \( X_A \) is a toric variety.
**$X_A$ and $Y_f$**

**Definition**

$X_A$ is defined as the Zariski closure of the following parameterization:

$$ (t^{a_0} : t^{a_1} : \ldots : t^{a_{n+1}}). $$

Then $X_A$ is a toric variety.

**Definition**

$Y_f$ is defined as the Zariski closure of the following parameterization:

$$ (f_0(x) : f_1(x) : \ldots : f_{n+1}(x)). $$

Then $Y_f$ is a curve.
The tropicalization of $X_A$ and $Y_f$ are easy to find, and since $Z_{A,f}$ is the Hadamard Product of $X_A$ and $Y_f$, we can find $\text{trop}(Z_{A,f})$. 
The tropicalization of $X_A$ and $Y_f$ are easy to find, and since $Z_{A,f}$ is the Hadamard Product of $X_A$ and $Y_f$, we can find $\text{trop}(Z_{A,f})$.

**Proposition**

Let $Z_{A,f}$ be an almost-toric hypersurface, then

$$\text{trop}(Z_{A,f}) = \{ u + \lambda \cdot v^T | u \in \text{row}(A), v \text{ is a column vector of } V_f, \lambda \geq 0 \}.$$
Here we sketch the proof: the tropicalization of $Z_{A,f}$ has maximal cells corresponding to $m$ rays. These cells also correspond to the edges of $\text{Newt}(Z_{A,f})$, in the following way such that every edge is orthogonal to the ray of the cell. ($P_A$ is skew-symmetric so for any column vector $v$ we have $v^T \cdot (P_A \cdot v) = 0$) Then we determine the directions of the edges of $\text{Newt}(Z_{A,f})$. 
Here we sketch the proof: the tropicalization of $Z_{A,f}$ has maximal cells corresponding to $m$ rays. These cells also correspond to the edges of $\text{Newt}(Z_{A,f})$, in the following way such that every edge is orthogonal to the ray of the cell. ($P_A$ is skew-symmetric so for any column vector $v$ we have $v^T \cdot (P_A \cdot v) = 0$). Then we determine the directions of the edges of $\text{Newt}(Z_{A,f})$.

To show that their lengths are the same as column vectors of $P_A \cdot V_f$, we need the notion of multiplicity of maximal cells. It is defined using commutative algebra. However, we use some results in [MS14] and find them using the index of lattices. Then we prove our main theorem.
For almost-toric hypersurfaces, our goal is: given the parameterization of an almost-toric hypersurface, to find its implicit polynomial $p(u_0, u_1, \ldots, u_{n+1})$. 
For almost-toric hypersurfaces, our goal is: given the parameterization of an almost-toric hypersurface, to find its implicit polynomial $p(u_0, u_1, \ldots, u_{n+1})$. Existing approaches include ideal elimination using Gröbner bases, which is inefficient when $n$ is large.
Based on the main theorem we have the following algorithm:

1. Compute $P_A$ from $A$, factorize $f_0, f_1, \ldots, f_{n+1}$ over $\mathbb{Q}$ into irreducible factors to get $V_f$.
2. Compute $P_A \cdot V_f$ and verify it has rank 2.
3. Find $\text{Newt}(Z_A, f)$ using our theorem.
4. Determine all possible monomials in variables $u_0, u_1, \ldots, u_{n+1}$ that could appear in the implicit polynomial.
5. Use linear algebra to compute the coefficients of these monomials.
In step (1) we can use mathematical softwares to factor the polynomials in $f_0, \ldots, f_{n+1}$. 
Step (1) and (2)

In step (1) we can use mathematical softwares to factor the polynomials in $f_0, \ldots, f_{n+1}$. 
By our theorem, the set of directed edges are the column vectors of $P_A \cdot V_f$. Then we need to arrange them in the correct order.
By our theorem, the set of directed edges are the column vectors of $P_A \cdot V_f$. Then we need to arrange them in the correct order. We could project these vectors to a 2-dimensional space, by choosing two of the coordinates $1 \leq c_1 < c_2 \leq n + 2$. There is still the problem of orientation: these directed edges admit two different arrangements. The correct orientation is determined by the sign of the $(P_A)_{c_1,c_2}$. 
Suppose we have arranged the column vectors of $P_A \cdot V_f$ in correct order, say $v_1, \ldots, v_m$. Then if we fix one point $c$ in $\mathbb{Z}^{n+2}$, we can add those vectors one by one to get all vertices of a polygon: $c, c + v_1, c + v_1 + v_2, \ldots, c + v_1 + v_2 + \ldots + v_{m-1}$, which is a translation of $\text{Newt}(\mathbb{Z}_A, f)$. 
Suppose we have arranged the column vectors of $P_A \cdot V_f$ in correct order, say $v_1, \ldots, v_m$. Then if we fix one point $c$ in $\mathbb{Z}^{n+2}$, we can add those vectors one by one to get all vertices of a polygon: $c, c + v_1, c + v_1 + v_2, \ldots, c + v_1 + v_2 + \ldots + v_{m-1}$, which is a translation of $\text{Newt}(Z_{A,f})$.

We claim that $\text{Newt}(Z_{A,f})$ is uniquely determined. Note that each vertex of $\text{Newt}(Z_{A,f})$ corresponds a monomial appeared in the implicit polynomial $p(u_0, u_1, \ldots, u_{n+1})$ and its $i$-th coordinate is the exponent of $u_{i-1}$ in the monomial. Since $p$ is irreducible, for each $i$ the minimum of these exponents must be 0. Then we know exactly how to translate our polygon to get $\text{Newt}(Z_{A,f})$. 
From $\text{Newt}(Z_A,f)$ we find all lattice points inside or on the boundary of this polygon. Thus we find all monomials that may appear in $p(u_0, u_1, \ldots, u_{n+1})$. 
Step (5)

Suppose we have undetermined coefficients for all possible monomials in $p$. After the substitution $u_i = t^{a_i} f_i(x)$ we can cancel the unique monomial in variables $t_1, \ldots, t_n$ and get a univariate polynomial in $x$ with undetermined coefficients. Since this polynomial is identically zero, the undetermined coefficients satisfy a system of homogeneous linear equations.
Next we use interpolation. Suppose there are $k$ possible monomials in the implicit polynomial, then we plug in $x$ by integers ranging from $-r$ to $r$, where $r = \lfloor \frac{k}{2} \rfloor$. Each interpolation gives a linear equation with $k$ coefficients. Then we use the `solve` command in Maple 17 to solve these coefficients. Since this is a homogeneous linear system, the solution space should be 1-dimensional. This leads us to add another equation, for example $a_1 = 1$ where $a_1$ is one of the coefficients, to guarantee the uniqueness of solution. After getting the solution, if all coefficients are rational, we normalize them so that their content is 1.
We then try some examples that both Macaulay 2 and Sagemath cannot solve in a reasonable time. We generate some samples as the input using the following method: let $n$ be the dimension of the torus, $d$ the degree of the homogeneous monomials and $k$ a positive integer. Then we choose $n + 2$ degree $d$ monomials randomly from all possible $\binom{n+d-1}{d}$ choices. For the univariate polynomials, we choose $n + 2$ polynomials of the form $(x - 2)^*(x - 1)^*x^*(x + 1)^*(x + 2)^*$, where each $*$ is a random integer between 0 and $k$. 
### Efficiency Test

<table>
<thead>
<tr>
<th>sample</th>
<th>degree</th>
<th># of terms</th>
<th>time cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>213</td>
<td>109</td>
<td>12.484s</td>
</tr>
<tr>
<td>2</td>
<td>109</td>
<td>80</td>
<td>1.594s</td>
</tr>
<tr>
<td>3</td>
<td>172</td>
<td>129</td>
<td>10.421s</td>
</tr>
<tr>
<td>4</td>
<td>474</td>
<td>275</td>
<td>156.969s</td>
</tr>
<tr>
<td>5</td>
<td>291</td>
<td>137</td>
<td>20.375s</td>
</tr>
<tr>
<td>6</td>
<td>179</td>
<td>97</td>
<td>8.110s</td>
</tr>
<tr>
<td>7</td>
<td>40</td>
<td>32</td>
<td>0.156s</td>
</tr>
<tr>
<td>8</td>
<td>27</td>
<td>14</td>
<td>0.140s</td>
</tr>
<tr>
<td>9</td>
<td>79</td>
<td>71</td>
<td>1.766s</td>
</tr>
<tr>
<td>10</td>
<td>281</td>
<td>148</td>
<td>20.719s</td>
</tr>
</tbody>
</table>

**Table:** $n = 4, d = 4, k = 5$
The table shows that the time needed to find the implicit polynomial of almost-toric hypersurfaces given by randomly generated inputs. Our algorithm improves the efficiency of finding the implicit polynomial of almost-toric hypersurfaces.
Reference

[MS14] D. Maclagan, B. Sturmfels,
Introduction to tropical geometry,
The End

Thank you!