1. Mark each statement True or False. Justify your answer precisely. (You can use any theorems or definitions you have learned in class or in the book. *Extra credit to those who answer all the true/false questions and justify them correctly.*)

   a. (3pts) For a matrix A, there exists a unique echelon form of A.

   b. (3pts) The rank of an $m \times n$ matrix A is exactly same as the number of pivot columns of the reduced echelon form of A.

   c. (3pts) Suppose that six vectors v_1, v_2, $\cdots$, v_6 satisfy :

   $$\{v_1, v_2, v_3, v_4\},\ \{v_3, v_4, v_5, v_6\},\text{ and } \{v_5, v_6, v_1, v_2\} \text{ are linearly independent sets of vectors.}$$

   Then, $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a linearly independent set.

d. (3pts) For every $n \times n$ matrix A, Col A is a subspace of $\mathbb{R}^n$.

e. (3pts) There exist two 3×3 matrices A and B such that

$$\text{Col } A \cup \text{Col } B$$

is a subspace of $\mathbb{R}^3$.

f. (3pts) For every 3×3 matrix A,

$$\text{Col } A \neq \text{Nul } A.$$

2. (6pts) Let A be the following 2×2 matrix:

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 5 \end{pmatrix}$$

Find all possible real numbers x such that

$$\det(A - xI) = 0$$

Here, I is the 2×2 identity matrix.

3. For the given matrix M

$$\begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 3 & 0 & -6 & 7 & -2 \\ 2 & 1 & 0 & 3 & 5 \\ 2 & 3 & 8 & 5 & 7 \end{pmatrix}$$

answer the two questions below.

a. (7pts) Find the reduced echelon form of M and a basis of Col M. Find m such that Col $M \subset \mathbb{R}^m$ and compute dim Col M.

b. (5pts) Define M_l as the matrix obtained from M by deleting the 5th column of M. Find M_l^{-1} and $\det M_l$.

4. Let A, B, C, and u be

$$A = \begin{pmatrix} 3 & 7 & -2 & 1 \\ 1 & 4 & 5 & 5 \\ 0 & 2 & 6 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 7 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 4 \\ -2 \\ 1 \\ -1 \end{pmatrix}$$

Compute the following :

a.(3pts) Au

b.(3pts) A^T

c.(4pts) BC^T

5. Let $\mathbb{P}_4$ be the set of all polynomials of degree at most 4. Define

$$S = \{p(x) \in \mathbb{P}_4 : p(1) = 0\}$$

a. (7pts) Show that S is a subspace of $\mathbb{P}_4$.

b. (7pts) Find a basis of S. What will $\dim S$ be?