1. a. (5pts) In $\mathbb{R}^5$, you are given 3 vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \; v_3 = \begin{pmatrix} 7 \\ -2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$$

Apply Gram-Schmidt (Orthogonalization) Process to find an orthonormal basis of $\text{Span}\{v_1, v_2, v_3\}$.

b. (8pts) Solve the least-squares problem

$$A\mathbf{x} = \mathbf{b} \text{ where } A = \begin{pmatrix} 1 & 3 & 7 \\ 1 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \; \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ -5 \\ 5 \end{pmatrix}$$

in two different ways. (Hint. One way is to use the result of **a.** For any theorems you might use, please state them correctly, though you do not need to prove the theorems.)

2. Define $\mathcal{B} = \left\{ \begin{pmatrix} 5 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$.

   a. (3pts) Find the inverse matrix of $P$ where

$$P = \begin{pmatrix} 5 & -3 & 1 \\ 5 & 2 & 2 \\ -3 & 0 & -1 \end{pmatrix}$$

   b. (4pts) Find the $\mathcal{B}$-coordinate of $\begin{pmatrix} 4 \\ 2 \\ -11 \end{pmatrix}$.

   c. (5pts) Let a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the map sending $\mathbf{x}$ to $A\mathbf{x}$ where

$$A = \begin{pmatrix} 12 & 15 & 40 \\ 10 & 17 & 40 \\ -6 & -9 & -22 \end{pmatrix}$$

   Find the $\mathcal{B}$-matrix for $T$.

3. Write "TRUE" if the statement is always true, "FALSE" if it is sometimes false. ***No** explanations are needed.*

   a. (2pts) Given a subspace $W$ of $V$, the orthogonal projection map from $V$ to $W$ is a one-to-one linear transformation.

   b. (2pts) The orthogonal complement of the null space of $A$ is the same as the column space of $A$ if $A$ is symmetric.

   c. (2pts) If the orthogonal complement of the null space of $A$ is the same as the column space of $A$, then $A$ is symmetric.

   d. (2pts) The quadratic form $Q$ on $\mathbb{R}^3$ defined as

   $$Q(x_1, x_2, x_3) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 - 4x_2x_3$$

   is an indefinite quadratic form.

   e. (2pts) Let a vector space $\mathbb{R}^3$ be equipped with an inner product defined as

   $$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = (x_1 + 2x_2)(y_1 + 2y_2) + x_3y_3$$

   In this inner product space, $(1, 0, 0)$ and $(0, 1, 0)$ are orthogonal still.

   f. (2pts) A square matrix $A$ is invertible if and only if 0 is not an eigenvalue of $A$.

4. (8pts) Find the maximum and minimum values of

$$Q(x_1, x_2, x_3) = -x_1^2 + x_2^2 - 4x_3^2 - 8x_1x_2 - 8x_2x_3$$

subject to the constraint

$$x_1^2 + x_2^2 + x_3^2 = 1$$

5. Let $A$ be

$$\begin{pmatrix} 3 & -4 & -4 \\ 2 & 1 & -4 \\ -2 & 0 & 5 \end{pmatrix}$$

whose characteristic polynomial $\chi_A(\lambda)$ is $-(\lambda-1)(\lambda-3)(\lambda-5)$.

a. (5pts) Find 3 linearly independent eigenvectors and, using them, find a diagonal matrix $D$ and an invertible matrix $P$ such that

$$P^{-1}AP = D$$

b. (6pts) You have found only one pair of $(D, P)$ in problem **a.** Find all possible $D$'s. For each $D$, find one corresponding invertible matrix $P$ such that $P^{-1}AP = D$.

6. Let $T$ be a transformation from $\mathbb{P}_2$ to $\mathbb{R}^3$ such that

$$T(\mathbf{p}(t)) = \begin{pmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \\ \mathbf{p}(2) \end{pmatrix}$$

a. (3pts) Show that $T$ is a linear transformation.

b. (6pts) Find $\ker T$. What is the dimension of $\ker T$? Conclude that im $T$ is a 3-dimensional subspace of $\mathbb{R}^3$ so that im $T = \mathbb{R}^3$. (*Regard $T$ as a linear transformation from $\mathbb{P}_2$ (3-dimensional vector space) to im $T$.*)

c. (6pts) Prove that $T$ is one-to-one and onto. And then interpret the fact as following :

A polynomial of degree at most 2 is **uniquely** determined by three points $(0, \mathbf{p}(0))$, $(1, \mathbf{p}(1))$, and $(2, \mathbf{p}(2))$.