EXAMPLE. Find the form for a particular solution to

$$y'' + 2y' - 3y = f(t),$$

where $f(t)$ equals
(a) $7\cos 3t$ (b) $2te^t \sin t$ (c) $t^2 \cos \pi t$ (d) $5e^{-3t}$ (e) $t^2 e^t$

1. An external force $F(t) = 2\cos 2t$ is applied to a mass-spring system with $m = 1$, $b = 0$, and $k = 4$, which is initially at rest; i.e., $y(0) = 0$, $y'(0) = 0$. Verify that $y(t) = \frac{1}{2}t\sin 2t$ gives the motion of this spring. What will eventually (as t increases) happen to the spring?

2. **Boundary Value Problems.** When the values of a solution to a differential equation are specified at two different points, these conditions are called **boundary conditions.** This excercise is to show that for boundary value problems there is no Existence-Uniqueness Theorem. Given that every solution to

$$y'' + y = 0$$

is of the form

$$y(t) = c_1 \cos t + c_2 \sin t,$$

where c_1 and c_2 are arbitrary constants, show that

(a) There is a unique solution to the given differential equation that satisfies the boundary condition $y(0) = 2$ and $y(\pi/2) = 0$.

(b) There is no solution to the given differential equation that satisfies $y(0) = 2$ and $y(\pi) = 0$.

(c) There are infinitely many solutions to the given differential equation that satisfy $y(0) = 2$ and $y(\pi) = -2$.

3. Prove the sum of angle formula for the sine function by following these steps. Fix x.

(a) Let $f(t) := \sin(x + t)$. Show that $f''(t) + f(t) = 0$, $f(0) = \sin x$, and $f'(0) = \cos x$.

(b) Use the auxiliary equation technique to solve the initial value problem $y'' + y = 0$, $y(0) = \sin x$, and $y'(0) = \cos x$.

(c) By uniqueness, the solution in part (b) is the same as $f(t)$ from part (a). Write this equality.