1 Quadratic Forms

1.1 Change of Variable in a Quadratic Form

Given any basis $\mathcal{B} = \{v_1, \cdots, v_n\}$ of $\mathbb{R}^n$, let $P = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$. Then,

$$\mathbf{y} = P^{-1}\mathbf{x}$$

is the $\mathcal{B}$-coordinate of $\mathbf{x}$. So, P, kind of, changes a variable into another variable.

Now, let A be a symmetric matrix and define a quadratic form $\mathbf{x}^T A\mathbf{x}$. Take P as the matrix of which columns are eigenvectors. (This can be done by the Spectral Theorem.) Especially, P has orthonormal column vectors. Hence,

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T (P^T AP)\mathbf{y}$$

and $P^T AP$ is a diagonal matrix of which diagonal entries are the eigenvalues of A. Such quadratic form can be written as

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

1.2 Classifying Quadratic Forms

1.2.1 Positive Definite

$$Q(\mathbf{x}) > 0 \text{ for all } \mathbf{x} \neq \mathbf{0}$$

Furthermore, if $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$, then Q is **positive semidefinite**.

1.2.2 Negative Definite

$$Q(\mathbf{x}) < 0 \text{ for all } \mathbf{x} \neq \mathbf{0}$$

Furthermore, if $Q(\mathbf{x}) \leq 0$ for all $\mathbf{x} \neq \mathbf{0}$, then Q is **negative semidefinite**.

1.2.3 Indefinite

$Q(\mathbf{x})$ has both positive and negative values.

In fact, the quadratic form is

positive definite if and only if the eigenvalues of A are all positive
negative definite if and only if the eigenvalues of A are all negative
indefinite if and only if A has both positive and negative eigenvalues.

Furthermore, it is *positive semidefinite* if and only if the eigenvalues of A are nonnegative, *negative semidefinite* if and only if the eigenvalues of A are nonpositive.

2 Constrained Optimization

2.1 Theorem 6 : Max and Min of $Q(\mathbf{x})$ given $||\mathbf{x}|| = 1$

Let A be a symmetric matrix that defines a quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. Then, under the condition that $\mathbf{x}$ is a unit vector, the maximum value of $Q(\mathbf{x})$ is the largest eigenvalue λ_{max} and the minimum value of $Q(\mathbf{x})$ is the smallest eigenvalue λ_{min}.

$$\max\{Q(\mathbf{x}) : ||\mathbf{x}|| = 1\} = \lambda_{max}, \quad \min\{Q(\mathbf{x}) : ||\mathbf{x}|| = 1\} = \lambda_{min}$$

2.2 Theorem 7, 8

Under the same assumptions as 2.1, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraints

$$\mathbf{x}^T\mathbf{x} = 1, \quad \mathbf{x}^T v_1 = 0$$

is the second largest eigenvalue λ_2.

1. Mark each statement True or False.

   a. The matrix of a quadratic form is a symmetric matrix.

   b. A positive definite quadratic form Q satisfies $Q(\mathbf{x}) > 0$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

   c. The expression $||\mathbf{x}||^2$ is a quadratic form.

   d. An indefinite quadratic form is either positive semidefinite or negative semidefinite.

   e. If A is symmetric and the quadratic form $\mathbf{x}^T A\mathbf{x}$ has only negative values for $\mathbf{x} \neq \mathbf{0}$, then the eigenvalues of A are all negative.

2. Show that if B is $m \times n$, then $B^T B$ is positive semidefinite; and if B is $n \times n$ and invertible, then $B^T B$ is positive definite.

3. Show that if an $n \times n$ matrix A is positive definite, then there exists a positive definite matrix B such that $A = B^T B$.

4. Let A and B be symmetric $n \times n$ matrices whose eigenvalues are all positive. Show that the eigenvalues of $A + B$ are all positive.

5. Let A be an invertible $n \times n$ symmetric matrix. Show that if the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite, then so is the quadratic form $\mathbf{x}^T A^{-1}\mathbf{x}$.