1 Symmetric Matrix A

A **symmetric matrix** is a matrix A such that $A^T = A$.

Its main diagonal entries are arbitrary, but other entries occur in pairs (on opposite sides of the main diagonal).

1.1 Theorem 1 : Orthogonal eigenvectors

If A is symmetric, then any two eigenvectors associated with distinct eigenvalues are orthogonal.

1.2 Theorem 2 : Orthogonally diagonalizable

An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.

1.3 The Spectral Theorem

An $n \times n$ symmetric matrix A has the following properties:

a. A has n real eigenvalues, counting multiplicities.

b. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.

c. The eigenspaces are mutually orthogonal.

d. A is orthogonally diagonalizable.

1.4 Spectral Decomposition

As a consequence of The Spectral Theorem, we can find orthonormal eigenvectors $v_1, \cdots, v_n$ and the corresponding eigenvalues $\lambda_1, \cdots, \lambda_n$ such that

$$\begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{pmatrix} A \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

So,

$$A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \cdots + \lambda_n v_n v_n^T$$

This representation of A is called a spectral decomposition.

2 Quadratic Forms

A **quadratic form** on $\mathbb{R}^n$ is a function $Q : \mathbb{R}^n \to \mathbb{R}$ such that there exists an $n \times n$ matrix satisfying

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^n$. The matrix A is called the **matrix of the quadratic form**.

Example. A quadratic form on $\mathbb{R}^2$ can be written as

$$Q(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + (b+c)xy + dy^2$$

So, every quadratic form on $\mathbb{R}^2$ can be written in the form of

$$Q(x, y) = ax^2 + bxy + cy^2$$

for some a, b, $c \in \mathbb{R}$.

---

1. Determine whether or not the matrix is orthogonal.

1) $\begin{pmatrix} .6 & .8 \\ .8 & -.6 \end{pmatrix}$ 2) $\begin{pmatrix} -5 & 2 \\ 2 & 5 \end{pmatrix}$

2. Mark each statement True or False. Justify your answer.

a. An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric.

b. An $n \times n$ symmetric matrix has n distinct real eigenvalues.

c. If $B = PDP^T$, where $P^T = P^{-1}$ and D is a diagonal matrix, then B is a symmetric matrix.

d. An orthogonal matrix is orthogonally diagonalizable.

3. Show that if A is an $n \times n$ symmetric matrix, then $(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$ for all $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^n$.

4. Suppose A is invertible and orthogonally diagonalizable. Explain why A^{-1} is also orthogonally diagonalizable.

5. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^2$.

a. $20x_1^2 + 15x_1x_2 - 10x_2^2$ b. x_1x_2

6. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.

a. $5x_1^2 - x_2^2 + 7x_3^2 + 5x_1x_2 - 3x_1x_3$

b. $x_3^2 - 4x_1x_2 + 4x_2x_3$