1 Similarity

Let A and B are $n \times n$ matrices. A and B are said to **be similar to** each other if there is an invertible matrix P

$$P^{-1}AP = B.$$

1.1 Diagonalization (revisit)

These two statements are the same.

A is diagonalizable.

and

There exists a diagonal matrix D such that A is similar to D.

1.2 Theorem 4

If $n \times n$ matrices A and B are similar, then they have the same *characteristic polynomial* and hence the same *eigenvalues.*

2 Linear Transformation (revisit)

We will discuss three examples. There is one new concept : **the matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$**. When $\mathcal{B} = \mathcal{C}$, we call the matrix as **the $\mathcal{B}$-matrix for T**.

EXAMPLE 1. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be bases for vector spaces V and W, respectively. Let $T : V \to W$ be a linear transformation with the property that

$$T(\mathbf{d}_1) = 3\mathbf{b}_1 - 3\mathbf{b}_2, \quad T(\mathbf{d}_2) = -2\mathbf{b}_1 + 5\mathbf{b}_2.$$

Find the matrix for T relative to $\mathcal{D}$ and $\mathcal{B}$.

EXAMPLE 2. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for a vector space V and let $T : V \to \mathbb{R}^2$ be a linear transformation with the property that

$$T(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3) = \begin{pmatrix} 2x_1 - 3x_2 + x_3 \\ -2x_1 + 5x_3 \end{pmatrix}.$$

Find the matrix for T relative to $\mathcal{B}$ and the standard basis for $\mathbb{R}^2$.

EXAMPLE 3. Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be the transformation that maps a polynomial $\mathbf{p}(t)$ into the polynomial $(t+3)\mathbf{p}(t)$.

a. Find the image of $\mathbf{p}(t) = 3 - 2t + t^2$.

b. Show that T is a linear transformation.

c. Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3\}$.

2.1 Theorem 8 : A role of eigenvectors

Suppose A is diagonalizable and $A = PDP^{-1}$. Then, for the basis $\mathcal{B}$ formed from the columns of P, then D is the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

3 Complex Eigenvalues

It is enough to solve Chapter 5.5 problems of Assignment 6. :)

4 Inner product

Recall that there are countless number of vector spaces other than $\mathbb{R}^n$. So are inner products. Even, there are countless inner products for one vector space. Let me first introduce the most common inner product on the vector space $\mathbb{R}^n$.

4.1 The most standard inner product space $\mathbb{R}^n$

Let's fix $n = 3$.

Every vector in $\mathbb{R}^3$ can be written of the form

$$(x_1, x_2, x_3).$$

We define the standard **inner product**, denoted by $\cdot$, on $\mathbb{R}^3$ in this way.

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = x_1y_1 + x_2y_2 + x_3y_3$$

Regarding each vector as a 3×1 matrix, then

$$u \cdot v = u^T v.$$

In this case, **the length of** v, denoted by $||v||$ is defined as the nonnegative square root of $v \cdot v$; $||v||^2 = v \cdot v$ and $||v|| \geq 0$.

Moreover, when a vector v has its length as 1, we say that v is a **unit vector**. The **distance between** u **and** v, written as $\text{dist}(u, v)$, is the length of the vector $u - v$: $\text{dist}(u, v) = ||u - v||$.

We can also show that u and v are perpendicular (or **orthogonal**) to each other if and only if $u \cdot v = 0$. This can be induced from the Pythagorian Theorem :

Two vectors u and v are orthogonal if and only if $||u + v||^2 = ||u||^2 + ||v||^2$.

In the last place, we define **ortogonality** for not only two vectors but also two sets of vectors in a very natural sense. Furthermore, given a subspace W of $\mathbb{R}^n$, we define the **orthogonal complement of** W, denoted by $W^\perp$, as the set of all vectors orthogonal to W.

4.2 Inner Product Space $(V, \langle \cdot, \cdot \rangle)$

As a function, an inner product on a vector space V is a map from $V \times V$ to $\mathbb{R}$, denoted by $\langle \cdot, \cdot \rangle$, satisfying below four axioms.

1) $\langle u, v \rangle = \langle v, u \rangle$ for all u and $v \in V$.

2) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all u, v, and $w \in V$.

3) $\langle cu, v \rangle = c\langle u, v \rangle$ for all u, $v \in V$, and $c \in \mathbb{R}$.

4) $\langle u, u \rangle \geq 0$ for all $u \in V$ and $\langle u, u \rangle = 0$ if and only if $u = 0 \in V$.

In words, you can say that a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ which is symmetric, coordinatewisely linear, and contains positiveness(?) is an inner product. A vector space equipped with an *inner product* is called an **inner product space**. Let's recall the definitions of many concepts in 4.1 above.

- **Norm of a vector** v : $||v|| = \sqrt{\langle v, v \rangle}$.
- **Unit vector** v : A vector v of length 1.
- **Distance between** u **and** v : $||u - v||$.
- **Orthogonal** u **and** v : $\langle u, v \rangle = 0$.
- **Orthogonal complement of** W : $W^\perp = \{v \in V : \langle v, w \rangle = 0 \text{ for all } w \in W\}$.

1. Show that if A and B are similar, then $\det A = \det B$.

2. Show that if A has n linearly independent eigenvectors, then so does A^T.

3. Define $T : \mathbb{P}_2 \to \mathbb{R}^3$ by

$$T(\mathbf{p}) = \begin{pmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{pmatrix}$$

   a. Find the image under T of $\mathbf{p}(t) = 5 + 3t$.

   b. Show that T is a linear transformation.

   c. Find the matrix for T relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$ and the standard basis for $\mathbb{R}^3$.

4. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ as the matrix transformation defined by $A = \begin{pmatrix} 4 & -2 \\ -1 & 5 \end{pmatrix}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

5. Let $A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, for $\mathbf{b}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$.

   a. Verify that $\mathbf{b}_1$ is an eigenvector of A but that A is not diagonalizable.

   b. Find the $\mathcal{B}$-matrix for T.

6. Show that if A is similar to B, then A^2 is similar to B^2.

7. Are those vectors orthogonal?

$$\begin{pmatrix} 12 \\ 3 \\ -5 \end{pmatrix} \quad \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$$

8. Mark each statement True or False. Justify your answer. (All vectors are in $\mathbb{R}^n$).

a. $v \cdot v = ||v||^2$.

b. For any scalar c, $u \cdot (cv) = c(u \cdot v)$.

c. If the distance from u to v equals the distance from u to $-v$, then u and v are orthogonal.

d. For a square matrix A, vectors in Col A are orthogonal to vectors in Nul A.

e. For any scalar c, $||cv|| = c||v||$.

f. If $||u||^2 + ||v||^2 = ||u + v||^2$, then u and v are orthogonal.

9. Verify this formula for vectors u and v in $\mathbb{R}^n$.

$$||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2$$

10. Show that if $\mathbf{x}$ in in both W and $W^{\perp}$, then $\mathbf{x} = \mathbf{0}$.