1 Change of Basis

1.1 Change of Basis in $\mathbb{R}^n$

Let $b_1 = \begin{pmatrix} -9 \\ 1 \end{pmatrix}$, $b_2 = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$, $c_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, $c_2 = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$. Find $P_{c \to b}$.

\[
\begin{pmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -9 & -5 \\ 0 & 7 & -35 & -21 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -9 & -5 \\ 0 & 1 & -5 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{pmatrix}.
\]

Hence, $P_{c \to b} = \begin{pmatrix} 6 & 4 \\ 5 & -3 \end{pmatrix}$.

1.2 Change of Basis in $\mathbb{P}_n$

Let $B = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ in $\mathbb{P}_2$. Find the change-of-coordinates matrix from the basis $B$ to the standard basis. Then write $t^2$ as a linear combination of the polynomials in $B$.

\[
\begin{align*}
\int -3t^2 &= 1 \cdot 1 + 0 \cdot t + (-3) t^2 \Rightarrow \text{first column} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \\
2 + t - 5t^2 &= 2 \cdot 1 + (1) t + (-5) t^2 \Rightarrow \text{second column} = \begin{pmatrix} 2 \\ -5 \end{pmatrix} \\
1 + 2t &= 1 \cdot 1 + 2 \cdot t + 0 \cdot t^2 \Rightarrow \text{third column} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\end{align*}
\]

We need to read $t^2$ using $B$.

\[
\begin{align*}
\left[ t^2 \right]_B &= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\
P_{c \to b} \cdot \left[ t^2 \right]_B &= \begin{pmatrix} 1 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \\ -2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 10 & -5 & 3 \\ -6 & 3 & -2 \\ -2 & 1 \end{pmatrix}
\end{align*}
\]

The standard basis for $\mathbb{P}_2$ is $\{1, t, t^2\}$.

2 Eigenvalues and Eigenvectors

From now on, we will only discuss about square matrices. Let $A$ be an $n \times n$ matrix. Assume that there exists a real number $\lambda$ such that

\[
\det(A - \lambda I) = 0.
\]

We call such a real number as an eigenvalue of $A$. If you remind the Invertible Matrix Theorem, it implies that $\text{Nul}(A - \lambda I)$ is not the zero space. Every nonzero vector $v \in \text{Nul}(A - \lambda I)$ is called an eigenvector.
1. If $A$ is a $7 \times 5$ matrix, what is the largest possible rank of $A$? If $A$ is a $5 \times 7$ matrix, what is the largest possible rank of $A$? Explain your answers.

5 and 5. From 2.9, we know that $\text{rank } A = \dim \text{Row } A = \dim \text{Col } A$. Hence, these maximal dimensions are 5.

2. Mark each statement True or False. Justify your answer.

a. The dimensions of the row space and the column space of $A$ are the same, even if $A$ is not square.

True. Because of 2.9, row operations do not change linear dependence relation among the rows of $A$.

b. If $B$ is the reduced echelon form of $A$, and if $B$ has three nonzero rows, then the first three rows of $A$ form a basis for Row $A$.

True. Because of 2.6, row operations preserve the linear dependence relations among the rows of $A$.

c. Row operations preserve the linear dependence relations among the rows of $A$.

True. In class, we did.

d. If $A$ and $B$ are row equivalent, then their row spaces are the same.

True. Row operations do not change the space.

3. Suppose $A$ is $m \times n$ and $b$ is in $\mathbb{R}^m$. What has to be true about the two numbers $\text{rank } (A \ b)$ and $\text{rank } A$ in order for the equation $Ax = b$ to be consistent?

\[ \text{rank } (A \ b) = \text{rank } A. \]

4. a. Is $\lambda = 2$ an eigenvalue of \( \begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix} \)?

\[ \det \left( \begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) = \det \left( \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \right) = 0 \quad \Rightarrow \quad \lambda = 2 \text{ is an eigenvalue of } \begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}. \]

b. Is \( \begin{pmatrix} -1 \\ 1 \end{pmatrix} \) an eigenvector of \( \begin{pmatrix} 5 & 2 \\ 3 & 6 \end{pmatrix} \)? If so, find the eigenvalue.

\[ \begin{pmatrix} 5 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} = 3 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \text{It is true and the eigenvalue is } 3. \]

c. Is \( \begin{pmatrix} 1 \\ -2 \end{pmatrix} \) an eigenvector of \( \begin{pmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{pmatrix} \)? If so, find the eigenvalue.

\[ \begin{pmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 13 \\ 1 \end{pmatrix} \neq \text{not a multiple of } \begin{pmatrix} -2 \\ 2 \end{pmatrix}. \quad \text{It is NOT true.} \]
5. Let \( \lambda \) be an eigenvalue of an invertible matrix \( A \). Show that \( \lambda^{-1} \) is an eigenvalue of \( A^{-1} \).

\[
(A^{-1} - \lambda I) \cdot \lambda A = (A I - A) \cdot \lambda A
\]

Therefore, \( \det(A^{-1} - \lambda I) = \det((A I - A) \cdot \lambda A) = \det(A I - A) \cdot \det(\lambda A) = 0 \). Therefore, \( \lambda^{-1} \) is an eigenvalue of \( A^{-1} \).

6. Show that if \( A^2 \) is the zero matrix, then the only eigenvalue of \( A \) is 0.

Suppose that \( Au = \lambda u \) for some scalar \( \lambda \) and \( u \) vector. 

Multiplying \( A \) on both sides of each side,

\( 0 = 0 \cdot u = A^2 u = \lambda A u = \lambda(A u) = \lambda^2 u \).

Hence, \( \lambda^2 = 0 \Rightarrow \lambda = 0 \).

7. Show that \( \lambda \) is an eigenvalue of \( A \) if and only if \( \lambda \) is an eigenvalue of \( A^T \).

\[
\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I)
\]

8. Let \( A \) be an \( n \times n \) matrix. Mark each statement True or False. Justify your answer.

a. If \( Ax = \lambda x \) for some vector \( x \), then \( \lambda \) is an eigenvalue of \( A \).

False. \( x \) should not be the zero vector.

b. A number \( c \) is an eigenvalue of \( A \) if and only if the equation \( (A - cI)x = 0 \) has a nontrivial solution.

True. \( Bx = 0 \) has a nontrivial sol'n if and only if \( \det B = 0 \).

c. If \( v_1 \) and \( v_2 \) are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

False. Counterexample. \((1, 0)\); \((1, 0)\) are eigenvectors and they are linearly independent, but have the same eigenvalue.

9. Construct an example of a \( 2 \times 2 \) matrix with only one distinct eigenvalue.

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

has only one distinct eigenvalue \( \lambda = 1 \).