1 One-to-one and Onto

Let A be an $m \times n$ matrix.

1.1 $\mathbf{x} \mapsto A\mathbf{x}$ is onto

a. A has a pivot position in every row.

b. The rank of A is m.

c. For every $\mathbf{b} \in \mathbb{R}^m$, there exists $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$.

d. Every $\mathbf{b}$ is a linear combination of the column vectors Ae_1, Ae_2, $\cdots$, Ae_n.

e. Col $A = \mathbb{R}^m$.

1.2 $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one

a. For every $\mathbf{b} \in \mathbb{R}^m$, there exists only one $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$ or there is no solution for the equation.

b. Every $\mathbf{b} \in \mathbb{R}^m$ has a unique representation

$$\mathbf{b} = c_1 A_1 + \cdots + c_n A_n$$

or there is no representation of that form.

2 Invertible matrix A

An $n \times n$ matrix A is invertible if and only if one of the followings is true

a. $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one and onto.

b. $A\mathbf{x} = \mathbf{0}$ has a unique solution $\mathbf{x} = \mathbf{0}$.

c. Nul $A = 0$.

d. rank $A = n$.

e. Col $A = \mathbb{R}^n$.

f. The reduced echelon form of A is the $n \times n$ identity matrix I.

2.1 Finding the inverse of a matrix A

There are two ways to find the inverse of a matrix A. What are they?

3 Vector Space

What does it mean by a set of vectors being linearly independent? or a basis?

How do we define the dimension of a vector space? Is it well-defined?

Practice Exam(?)

1. Prove that Col A is a vector space indeed. You can prove through 3 steps.

   a. Explain why the zero vector is in Col A.

   b. Show that the vector $A\mathbf{x} + A\mathbf{w}$ is in Col A.

   c. Given a scalar c, show that $c(A\mathbf{x})$ is in Col A.

2. Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{pmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{pmatrix}$.

   a. Show that T is a linear transformation.

   b. Find a polynomial $\mathbf{p}$ in $\mathbb{P}_2$ that spans the kernel of T, and describe the range of T.

3. Define a linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{pmatrix} \mathbf{p}(0) \\ \mathbf{p}(0) \end{pmatrix}$. Find polynomials $\mathbf{p}_1$ and $\mathbf{p}_2$ in $\mathbb{P}_2$ that span the kernel of T, and describe the range of T.

4. Explain what is wrong with the following discussion: Let $\mathbf{f}(t) = 3 + t$ and $\mathbf{g}(t) = 3t + t^2$, and note that $\mathbf{g}(t) = t\mathbf{f}(t)$. Then $\{\mathbf{f}, \mathbf{g}\}$ is linearly dependent because $\mathbf{g}$ is a multiple of $\mathbf{f}$.

5. Let $M_{2\times 2}$ be the vector space of all 2×2 matrices, and define $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A + A^T$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

   a. Show that T is a linear transformation.

   b. Let B be any element of $M_{2\times 2}$ such that $B^T = B$. Find an A in $M_{2\times 2}$ such that $T(A) = B$.

   c. Show that the range of T is the set of B in $M_{2\times 2}$ with the property that $B^T = B$.

   d. Describe the kernel of T.

6. Given subspaces H and K of a vector space V, the sum of H and K, written as $H + K$, is the set of all vectors in V that can be written as the sum of two vectors, one in H and the other in K; that is,

$$H + K = \{\mathbf{w} : \mathbf{w} = \mathbf{u} + \mathbf{v} \text{ for some } \mathbf{u} \in H \text{ and some } \mathbf{v} \in K\}$$

   a. Show that $H + K$ is a subspace of V.

   b. Show that H is a subspace of $H + K$ and K is a subspace of $H + K$.

7. Mark each statement True or False. Justify each answer.

a. The set of all linear combinations of $v_1, \cdots, v_p$ is a vector space.

b. If $\{v_1, \cdots, v_{p-1}\}$ spans V, then $\{v_1, \cdots, v_{p-1}, v_p\}$ spans V.

c. Row operations on a matrix A can change the linear dependence relations among the columns of A.

d. Row operations on a matrix can change the null space.

e. Row operations on a matrix can change the column space.

f. If B is obtained from a matrix A by several elementary row operations, then rank B =rank A.

g. If A is $m \times n$ and rank $A = m$, then the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

8. Let H be an n-dimensional subspace of an n-dimensional vector space V. Explain why $H = V$.