1 Origin of the Determinants

1.1 Linear Independence and Nonzero Volume

When do two vectors in $\mathbb{R}^2$ make a parallelogram with nonzero volume?

When is a set of two vectors of $\mathbb{R}^2$ linearly independent?

You have to have the same answers for those two problems. What if we change two into n and 2 into n? We still have the same answer intuitively.

Hence, determining 'linear independence' of a set of n vectors is closely related to computing 'the volume' of an n-dimensional parallelepiped.

1.2 A linearity property of Volume

As an easy example, let's think about two 2-dimensional vectors. What properties does the volume of two vectors have as those two vectors vary? In some sense, the volume is linear. Here are three properties the volume has to have.

a) Normalized volume

b) Zero volume condition

c) Linearity

One can prove that there exists a unique volume function that maps n n-dimensional vectors to its volume and this can be the definition of the ***determinant***.

1.3 Why is the *determinant of a matrix* important?

Because a square matrix A is invertible if and only if $\det A \neq 0$. So, the determinant of A determines invertibility of a square matrix A.

2 Definition of Determinants (Explicit one)

Note. The ***determinant of a matrix*** can be only defined for a **square matrix** A.

2.1 The determinant of a 2×2 matrix

Given a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we define the ***determinant of*** A, denoted by $\det A$, as following :

$$\det A = ad - bc$$

2.2 The determinant of a 3×3 matrix

Given a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

we define the ***determinant of*** A, also denoted by $\det A$, as following :

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

2.3 The determinant of a $n \times n$ matrix

There is an explicit form for $\det A$. It is

$$\det A = \sum_{\sigma \,:\, a\ permutation} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

where

$$A = (a_{ij}),$$

that is, the entry on ith row and jth column is a_{ij}. However, in order to explain what *a permutation* σ and $\operatorname{sgn}(\sigma)$ mean for σ, we need many things to learn.

In the textbook, there is a definition of the *determinant of* A using submatrices and defined inductively. Let A_{ij} be the submatrix of A formed by deleting the ith row and jth column of A. Then,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}.$$

Apply the definition when $n = 3$.

2.4 How does *elementary row operations* affect to the determinant of A?

Recall that there are three *elementary row opertaions* ; replacement, interchange, and scaling.

2.5 What other properties does the determinant have?

$$\det A^T = \det A \quad \text{and} \quad \det AB = \det A \cdot \det B$$

1. Compute the determinants of each matrices.

   a. $\begin{pmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{pmatrix}$

   b. $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{pmatrix}$

   c. $\begin{pmatrix} 5 & -2 & 4 \\ 0 & 3 & -5 \\ 2 & -4 & 7 \end{pmatrix}$

2. Let $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$. Write $5A$. Is $\det 5A = 5 \det A$?

3. Mark each statement True or False. Justify each answer.

   a. A row replacement operation does not affect the determinant of a matrix.

   b. The determinnat of A is the product of the pivots in any echelon form U of A, multipled by $(-1)^r$, where r is the number of row interchanges made during row reduction from A to U.

   c. If the columns of A are linearly dependent, then $\det A = 0$.

   d. $\det(A + B) = \det A + \det B$.

   e. If $\det A$ is zero, then two rows or two columns are the same, or a row or a column is zero.

   f. If AB is invertible, then A is invertible.

4. Show that if A is invertible, then $\det A^{-1} = \frac{1}{\det A}$.

5. Let U be a square matrix such that $U^T U = I$. Show that $\det U = \pm 1$.

6. Suppose that A is a square matrix such that $\det A^4 = 0$. Explain why A cannot be invertible.

7. Verify that $\det A = \det B + \det C$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & u_1 + v_1 \\ a_{21} & a_{22} & u_2 + v_2 \\ a_{31} & a_{32} & u_3 + v_3 \end{pmatrix}, \quad B = \begin{pmatrix} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ a_{31} & a_{32} & u_3 \end{pmatrix}, \quad C = \begin{pmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{pmatrix}$$

Note, however, that A is not the same as $B + C$.

8. Define $T : \mathbb{R}^4 \to \mathbb{R}^4$ as following : Given $(a, b, c, d) \in \mathbb{R}^4$, we can define a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and then let T send A to $A + A^T$. Prove that this transformation is linear.