1 Subspaces of $\mathbb{R}^n$

1.1 Which set is called called a subspace of $\mathbb{R}^n$?

Given a set of vectors $\{v_1, \cdots, v_n\}$, we have defined Span$\{v_1, \cdots, v_n\}$. This gives an exmaple of a subspace!

1.2 Basis for a subspace

For a given subspace H of $\mathbb{R}^n$, a ***basis*** for H is a *linearly independent set* in H that spans H, that is, Span(the set)$= H$.

1.3 Dimension of a subspace

For a given nonzero subspace H, we denote dim$H$ as the ***dimension*** of the number of vectors in any basis for H. In order to define the concept of the *dimension* we need to prove that the number of vectors in two different bases for H are always same. How?

2 Column space and Null space of a Matrix

Column Space of a matrix A (Col A):

Null Space of a matrix A (Nul A):

Given a $m \times n$ matrix A, the column space of A and the null space of A are both examples of subspaces. Why?

2.1 Rank of a matrix

The ***rank*** of a matrix A, denoted by rank A, is defined as the *dimension* of the column space of A. We have one important theorem named **The Rank Theorem**;

$$\text{rank } A + \dim \text{Nul } A = n, \quad \text{where } A \text{ is an } n \times n \text{ matrix}$$

Exercise 1. Determine which sets below are bases for $\mathbb{R}^2$ or $\mathbb{R}^3$. Justify your answer.

a) $\begin{pmatrix} 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 16 \\ -3 \end{pmatrix}$

b) $\begin{pmatrix} -2 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ -10 \end{pmatrix}$

c) $\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$

d) $\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$

e) $\begin{pmatrix} 3 \\ -8 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \\ -5 \end{pmatrix}$

Exercise 2. If B is a 7×7 matrix and Col $B = \mathbb{R}^7$, what can be said about solutions of equations of the form $B\mathbf{x} = \mathbf{b}$ for $\mathbf{b} \in \mathbb{R}^7$?

Exercise 3. Suppose a 4×6 matrix A has four pivot columns. Is Col $A = \mathbb{R}^4$? Is Nul $A = \mathbb{R}^2$? Explain your answers.

Exercise 4. Suppose a 4×7 matrix A has three pivot columns. Is Col $A = \mathbb{R}^3$? What is the dimension of Nul A? Explain your answers.

Exercise 5. Justify each answer.

a) There exists a 3×5 matrix A such that dim Nul $A = 3$ and dim Col $A = 2$.

b) Show that a set $\{v_1, \cdots, v_5\}$ in $\mathbb{R}^n$ is linearly independent if dim Span$\{v_1, \cdots, v_5\} = 4$.

c) Let A be an $n \times p$ matrix whose column space is p-dimensional. Explain why the columns of A must be linearly independent.