0.1 Matrix Multiplication (revisit)

Suppose that A is an $m \times n$ matrix and B a $k \times l$ matrix. When is AB (the multiplication of A and B) well-defined?

1 The Inverse of a matrix

In order to talk about the inverse of a matrix A, we require A be an $n \times n$ matrix, that is, the number of rows should be same as the number of columns. However, not all of $n \times n$ matrices have its inverse. When A has its (unique) inverse, we say that A is ***invertible***.

1.1 2×2 matrix A

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has its inverse when $ad - bc$ is nonzero and the inverse matrix A^{-1} is

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

1.2 How to solve $A\mathbf{x} = \mathbf{b}$

1.3 Some properties of inverse matrices

Given two *invertible* matrices A and B. Let A^{-1} and B^{-1} be their inverse matrices. Then the following formulas hold ;

$$(A^{-1})^{-1} = A \tag{1}$$
$$(AB)^{-1} = B^{-1}A^{-1} \tag{2}$$
$$(A^T)^{-1} = (A^{-1})^T \tag{3}$$

1.4 Theorem 7

An $n \times n$ matrix A is *invertible* if and only if A is *row equivalent* to I_n (the identity matrix).

1.5 An algorithm for finding A^{-1}

1.6 The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent.

1) A is *invertible*
2) A is *row equivalent* to the $n \times n$ identity matrix
3) A has n pivot positions
4) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
5) The columns of A form a linearly independent set
6) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
7) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$
8) The columns of A span $\mathbb{R}^n$
9) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
10) There is an $n \times n$ matrix C such that $CA = I$
11) There is an $n \times n$ matrix D such that $AD = I$
12) A^T is *invertible*

1. Compute each matrix sum or product if it is defined. If an expression is undefined, explain why. Let

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 5 \\ -1 & 4 \end{pmatrix}, \quad E = \begin{pmatrix} -5 \\ 3 \end{pmatrix}.$$

   $B - 2A$, AC, CD, $2C - 3E$, DB, EC.

2. If a matrix A is 5×3 and the product AB is 5×7, what is the size of B?

3. Find the inverses of the matrices of
$$\begin{pmatrix} 8 & 6 \\ 5 & 4 \end{pmatrix}, \qquad \begin{pmatrix} 2 & -4 \\ 4 & -6 \end{pmatrix}$$

4. Use matrix algebra to show that if A is invertible and D satisfies $AD = I$, then $D = A^{-1}$.

5. Explain why the columns of an $n \times n$ matrix A are linearly independent when A is invertible.

6. If $n \times n$ matrices E and F have the property that $EF = I$, then E and F commute. Explain why.