How to write down right proofs exactly

Find the limit, if it exists, or show that the limit does not exist.

a) \[ \lim_{{(x,y) \to (0,0)}} \frac{xy^2}{x^2 + y^4} \]

**Proof that the limit does not exist**

If \((x, y)\) approaches to \((0, 0)\) along the curve \(x = y^2\), the value of the function is constantly

\[
\frac{y^2 y^2}{y^4 + y^4} = \frac{1}{2}.
\]

However, if we send \((x, y)\) to \((0, 0)\) along the curve \(x = 2y^2\),
the value of the function is constantly

\[
\frac{2y^2 y^2}{4y^4 + y^4} = \frac{2}{5}.
\]

Hence, the value of \(f(x, y) = \frac{xy^2}{x^2 + y^4}\) depends on the way by which \((x, y)\) approaches to \((0, 0)\). Therefore, it has no limit.

b) \[ \lim_{{(x,y) \to (0,0)}} \frac{x^2 y^2}{x^2 + y^2} \]

**Proof that the limit exists and is 0.**

Using the inequality

\[
x^2 + y^2 \geq 2|xy|,
\]

we get

\[
0 \leq \left| \frac{x^2 y^2}{x^2 + y^2} \right| \leq \frac{x^2 y^2}{2|xy|} = \frac{1}{2}|xy|.
\]

We know that \( \frac{1}{2}|xy| \) is continuous on the whole plane. Hence, taking \( \lim \) on the inequality above, we get

\[
0 \leq \lim_{{(x,y) \to (0,0)}} \left| \frac{x^2 y^2}{x^2 + y^2} \right| \leq \lim_{{(x,y) \to (0,0)}} \frac{1}{2}|xy| = \frac{1}{2} \cdot 0 = 0
\]

This shows that \[ \lim_{{(x,y) \to (0,0)}} \frac{x^2 y^2}{x^2 + y^2} = 0. \]