MATH 1B: CALCULUS DISCUSSION SECTION 2: HAPPY HALLOWEEN! **SOLUTIONS**

Some Scary Series:
First, determine what test you would use to prove the following series are convergent or divergent. Then, go back and prove your hypothesis. (* indicates that you should find the sum of the series).

(a) * $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^{n+1}}$ First, we show this converges using the n^{th} root test.

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \left(\frac{1}{n2^{n+1}}\right)^{1/n} \\ &= \lim_{n\to\infty} \frac{1}{n^{1/n}2^{1+1/n}} \\ &= \frac{1}{2}\end{aligned}$$

Where we have used that $\lim n^{1/n} = 1$ (Make sure you know how you would prove this!). Hence the series converges by the n^{th} root test. Now for the more exciting part: figuring out what value this sum converges to. The key here is to think of this series as a power series representation of a function $f(x)$ which has been evaluated at some point x. If we define $f(x) = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n}$ then we see that $f(-1/2)$ is the quantity we are interested in. The game now is to find a simple (closed form) representation for $f(x)$. This is not too difficult if we scratch our heads a bit and notice that this looks a heck of a lot like a geometric series, except for that pesky factor of n. How do we account for that? Well, we learned in Math 1A that $\int x^{n-1}dx = \frac{1}{n}x^n$. Except, curses! We have too many powers of x! How do we deal with that? Well, pull one out:

$$\frac{x^{n+1}}{n} = x\frac{x^n}{n} = x\int x^{n-1}dx$$

So we can write our function as

$$\begin{aligned}f(x) &= \sum_{n=1}^{\infty} x\int x^{n-1}dx \\ &= x\sum_{n=1}^{\infty}\int x^{n-1}dx \\ &= x\int\sum_{n=1}^{\infty} x^{n-1}dx \\ &= x\int\sum_{n=0}^{\infty} x^{n}dx\end{aligned}$$

Now we recognize the expression for the good 'ol geometric series, substitute in, and get a nice expression for f:

$$f(x) = x\int\frac{1}{1-x} = -x\ln(1-x)$$

Now we simply plug in $-1/2$ to find that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^{n+1}} = \frac{1}{2}\ln(\frac{3}{2})$$

(b) $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{3^{2n}}$ Use the n^{th} root test here to find that the series (which is centered, of course, about $x=0$ has radius of convergence $R=3$; plug in the endpoints to find that it converges when $|x|<3$.

(c) $\sum_{n=1}^{\infty}(-1)^n \tan^{-1}\left(\frac{1}{n}\right)$ Here we use our intuition that $\tan^{-1}(1/n)$ gets small like $1/n$ to motivate a limit comparison test with the divergent series $\sum_{n=1}^{\infty} 1/n$.[1] Thus we examine the limit of the ratio of the terms of there series:

$$\lim_{n\to\infty} \frac{\tan^{-1}(1/n)}{1/n}$$

Considering now a function of a continuous variable and using L'Hopital's rule to evaluate the limit, we find that

$$\lim_{x\to\infty} \frac{\tan^{-1}(x)}{1/x} = \lim_{x\to\infty} \frac{\frac{1}{1+x^2}\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}} = 1$$

Since this limit is 1, but the limit comparison test either both series diverge or both series converge. The harmonic series is divergent (if you remember one thing about series 20 years from now, I suggest this fact!) and hence the series in question is also divergent.

(d) * $\sum_{n=0}^{\infty}\left(\frac{1}{1+3\cdot(-1)^n}\right)^n$ This series looks pretty horrifying, but notice that we really have two sums here:

$$\sum_{n=0,\ n \text{ even}}^{\infty}\left(\frac{1}{1+3}\right)^n + \sum_{n=1,\ n \text{ odd}}^{\infty}\left(\frac{1}{1-3}\right)^n$$

where we are allowed to split the series up into two pieces because each piece is easily recognized as a convergent geometric series. Rewriting again, we have

$$\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{2n} + \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{2n+1}$$

Which allows us to find

$$\frac{1}{1-(\frac{1}{4})^2} - \frac{1}{2}\cdot\left(\frac{1}{1-(\frac{1}{2})^2}\right)$$

(e) * $\sum_{n=1}^{\infty}\frac{(-1)^n\pi^{2n}}{n!}$ Okay, now we have a series with a GASP! *factorial* in it (!). What shall we do? Well, this reminds us a rather lot of our favorite Taylor series (or at least it's MY favorite Taylor series):

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

so then let

$$f(x) = e^x - 1 = \sum_{n=0}^{\infty}\frac{x^n}{n!} - 1 = \sum_{n=1}^{\infty}\frac{x^n}{n!}$$

[1] You may not have an intuition at this point for the fact that "$\tan^{-1}(1/n)$ gets small like $1/n$." That is okay. Another strategy here is to think, well, this thing has a 1/n in it and is not OBVIOUSLY divergent since it does not fail the test for divergence, so maybe it is SNEAKILY divergent like the series 1/n, so what the hell, let's TRY a comparison with 1/n.

Which immediately allows us to write

$$f(-\pi^2) = \sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n}}{n!}$$

Hence our series is equal to $e^{-\pi^2} - 1$. In retrospect, this series is convergent because the Taylor series for e^x has a radius of convergence of ∞. We could have proven this earlier with a quick ratio test.

(f) $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^{0.3}}$ Using our by now rather well-honed intuition about series of this form, we try a limit comparison test with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{1.3}}$. Taking the limit of the ratio of the terms, we have

$$\lim_{n\to\infty} \frac{\frac{\sin(1/n)}{n^{0.3}}}{\frac{1}{n^{1.3}}} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$$

Where the last step may be proved using L'Hopital's Rule. Hence both series are divergent by the limit comparison test.

(g) $\sum_{n=1}^{\infty} ne^{-n^2}$ The terms of this series are positive and decreasing, so we let $f(x) = xe^{-x^2}$ and apply the integral test to decide that the series converges.

(h) $\sum_{n=2}^{\infty} \frac{n}{(\ln(n))^{\ln(n)}}$ Re-write the n^{th} term as

$$\frac{n}{\ln(n)^{\ln(n)}} = \frac{e^{\ln(n)}}{e^{\ln(\ln(n))\cdot\ln(n)}} = \frac{1}{e^{\ln(n)\cdot(\ln(\ln(n))-1)}} = \frac{1}{n^{(\ln(\ln(n))-1)}}$$

which appears to be a tragic mess until we think a bit and realize that $\ln(\ln(n))$ is a function which grows (albeit VERY slowly) with n. Thus if we pick n large enough,$\ln(\ln(n)) > 3$, [2] we have $a_n < 1/n^2$ for n bigger than some number which we will call N. Since we don't care about the first couple of hundred million terms (they may add up to something BIG, but it's still finite!) the important part is the tail end of the sum, $\sum_{n=N}^{\infty} a_n$, which is convergent by comparison with the convergent p-series $\sum_{n=N}^{\infty} 1/n^2$.

(i) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^3}$ Use the n^{th} root test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n^2} = \lim_{n\to\infty} b_n$$

Let $c_n = \ln(b_n)$ and compute

$$\begin{aligned}
\lim_{n\to\infty} c_n &= \lim_{n\to\infty} n^2 \ln(\frac{1}{1+1/n}) \\
&= \lim_{n\to\infty} \frac{-\ln(1+1/n)}{\frac{1}{n^2}} \\
&= \underbrace{\lim_{n\to\infty} \frac{\frac{1}{1+1/n}\cdot\frac{-1}{n^2}}{\frac{-2}{n^3}}}_{\text{by L'Hoptials}} \\
&= -\infty
\end{aligned}$$

Since $\lim_{n\to\infty} \ln(b_n) = -\infty$, we have $\lim_{n\to\infty} b_n = 0$, and hence the series converges by the ratio test.

[2] Specifically, we need $n > e^{e^3} \approx 300,000,000$

(j) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ Use a limit comparison test with our favorite divergent series, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n$, using L'Hoptial's rule to evaluate the resultant limits. We find that $\lim_{n\to\infty} a_n/b_n = 1$, so the series is divergent.

(k) $\sum_{n=1}^{\infty}(-1)^n \left(\sin(1/n^2)\right)^{1/3}$ This series satisfies the conditions for the alternating series test (the terms are decreasing and have a limit which is zero), so we find that it converges conditionally. To check absolute convergence, do a limit comparison test with $b_n = 1/n^{2/3}$.

(l) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$
"Rationalize" the terms to write the series as

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n} \cdot \left(\frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}}\right) = \sum_{n=1}^{\infty} \frac{2}{n\sqrt{n+1}+\sqrt{n-1}}$$

Which motivates a limit comparison test with $b_n = 1/n^{3/2}$.

(m) $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n+5}$ The series converges conditionally (check that it satisfies conditions for the AST!) the absolute values diverge, as can be shown by a limit comparison test with the series $\sum_{n=1}^{\infty} 1/n^{1/2}$.

REMARK: Other strategies include:

1. Checking if a sum is a telescoping series (i.e. $\sum_{n=1}^{\infty} \tan^{-1}(n+1) - \tan^{-1}(n) = \pi/4$).
2. Using the ratio test is almost always a good idea if you see factorials.
3. We used LCT a lot above but it can also be very useful to use a comparison test where you find a series whose terms are always larger or smaller than those of the series in question. For example, $\sum_{n=1}^{\infty} \frac{1}{n^{7/2}+5} < \sum_{n=1}^{\infty} \frac{1}{n^{7/2}}$, and $1/n^{7/2}$ is a convergent p- series.
4. IT IS VERY IMPORTANT TO MAKE SURE YOU KNOW WHAT THE PROBLEM IS ASKING YOU TO DO! Are you deciding if a sum is absolutely convergent? Conditionally convergent? Divergent? For what values of x or p the series is convergent or divergent? What the sum of the series is? Read carefully and it will save you a lot of headache.

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