Series Manipulations

Here's a (partial) list of some manipulations you can perform on series and what they modify

- Re-indexing
  - Changes: starting place, form of a_n
  - Doesn't change: value
  - Reminders: Don't forget about "hidden" parentheses around each n. When you decrease the starting place, you have to **increase** every n in a_n by the same amount.
  - Example: $\displaystyle\sum_{n=3}^{\infty} \frac{nx^{2n+1}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(n+3)x^{2(n+3)+1}}{(2(n+3))!} = \sum_{n=0}^{\infty} \frac{(n+3)x^{2n+7}}{(2n+6)!}$
- Factoring in/out of the $\sum$
  - Changes: form of a_n
  - Doesn't change: starting place, value
  - Reminders: You can **only** factor out things that don't involve an n. However, as far as the sum is concerned, x is a constant and can be moved in/out.
  - Example: $\displaystyle\frac{1}{8}\sum_{n=0}^{\infty} n^2 \frac{x^{n+2}}{2^{2n-3}} = x^2 \sum_{n=0}^{\infty} n^2 \frac{x^n}{2^{2n}} = x^2 \sum_{n=0}^{\infty} n^2 \left(\frac{x}{4}\right)^n$
- Adding/subtracting missing terms
  - Changes: starting place
  - Doesn't change: form of a_n, value
  - Reminders: Make sure you remember **all** the extra terms and that $x^0 = 0! = 1$
  - Example: $\displaystyle\sum_{n=4}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{3} \frac{x^n}{n!} = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) - \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)$

Finding Sums

Finding sums of series is somewhat of an art, but here's a general outline of the procedure I use when I'm stuck:

1. Figure out which known series you're aiming for. The denominator is usually a good indication, but this is definitely where the "art" comes in. If there's a factorial in the denominator, I'd start with whatever series matches that.
2. Figure out what your "x" is going to be. Numbers raised to an n are probably going to end up being part of the x you plug in at the end. Depending on your which known Maclaurin Series you're using, sometimes this includes the $(-1)^n$ part and sometimes it doesn't.
3. Figure out how to get from the known series to the one you have. Here's some tricks:
   - An extra n terms in the numerator usually comes from a derivative of x^n, x^{2n}, x^{2n+1}, etc.
   - Extra n terms in a denomintor usually come from integrals
   - If your series starts in the wrong place, but you like the form of a_n, use the third manipulation from above to fix it
   - Don't forget about your exponent laws: $\sum x^{3n+2} = x^2 \sum \left(x^3\right)^n$
   - One thing that's helpful is to learn to think relatively. For example, if you see $(n+1)x^n$, you should think of this as "a power of x multiplied by the number one bigger than the exponent," which is $(x^{n+1})'$
4. Start with the known series from the Maclaurin Series sheet and its function equivalent and perform the manipulations from step 2 to both sides. A few more tricks:
   - If you have an x^{n-1} but want to make an n show up, you can multiply both sides by x and **then** take the derivative. Similar tricks work for denominators from integrals.
   - Make sure you do things in the appropriate order to both sides! Multiplying by x and then differentiating is very different then differentiating and then multiplying by x.
   - If you ever integrate, remember to include a $+C$ and then solve for it.
5. Plug in the appropriate value of x into both sides.