MATH 1B: CALCULUS DISCUSSION SECTION 2: QUIZ 9

Please write neatly and show all your work.

1. (2 points) Determine the radius of convergence for series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$$

and show $y = f(x)$ is a solution of the differential equation

$$y' = 1 + xy$$

To determine the radius of convergence, use the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{\frac{x^{2(n+1)+1}}{1\cdot 3\cdots 2(n+1)+1}}{\frac{x^{2n+1}}{1\cdot 3\cdots(2n+1)}} \\
&= \lim_{n\to\infty} \frac{x^{2n+3}}{1\cdot 3\cdots(2n+1)(2n+3)} \frac{1\cdot 3\cdots(2n+1)}{x^{2n+1}} \\
&= \lim_{n\to\infty} \frac{x^2}{(2n+3)} \\
&= 0
\end{aligned}$$

The series is absolutely convergent for all x by the ratio test.
Now, differentiate the power series to find that

$$y' = \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{1\cdot 3\cdots(2n+1)} = \sum_{n=0}^{\infty} \frac{x^{2n}}{1\cdot 3\cdot(2n-1)}$$

The right hand side of the differential equation is

$$1 + xy = 1 + x\sum_{n=0}^{\infty} \frac{x^{2n+1}}{1\cdot 3\cdots(2n+1)} = 1 + \sum_{n=0}^{\infty} \frac{x^{2n+2}}{1\cdot 3\cdot(2n+1)}$$

now reindex, letting $k = n+1$:

$$1 + xy = 1 + \sum_{k=1}^{\infty} \frac{x^{2k}}{1\cdot 3\cdots(2k-1)}$$

Noticing that the $k = 0$ term is equal to 1, we rewrite it as one sum starting at zero to obtain the same expression as we had for y':

$$1 + xy = \sum_{k=0}^{\infty} \frac{x^{2k}}{1\cdot 3\cdots 2k-1} = y'$$

Remark: This was a problem on Persson's second midterm from 2009.

2. (3 points) Find the equation for a curve $y(x)$ which at every point (x, y) has a slope equal to $(1+y^2)\sec^2(x)$ and which passes through the point $(0, \pi)$.
This is a differential equation of the form:

$$\frac{dy}{dt} = (1+y^2)\sec^2(x) \quad y(0) = \pi$$

It is seperable:

$$\int \frac{dy}{1+y^2} = \int \sec^2(x)dx$$
$$\tan^{-1}(y) = \tan(x) + c$$

The initial condition gives

$$\tan^{-1}(\pi) = \tan(0) + c$$

So we have that $c = \tan^{-1}(\pi)$ Thus we have a solution

$$y = \tan(\tan(x) + \tan^{-1}(\pi))$$

3. (3 points) Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(n+1)}{4^n}$$

Rewrite this number as a power series evaluated at a point, then figure out what function the power series came from. Write

$$f(x) = 4\sum_{n=0}^{\infty}(n+1)x^{n+1}$$

because we then have that

$$f(-1/4) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(n+1)}{4^n}$$

So we just need to find a simple expression for $f(x)$ and evaluate it at $-1/4$. We recognize that $(n+1)x^{n+1} = x\frac{d}{dx}x^{n+1}$, so

$$f(x) = 4\sum_{n=0}^{\infty} x\frac{d}{dx}x^{n+1}$$
$$f(x) = 4x\frac{d}{dx}\sum_{n=0}^{\infty} x^{n+1}$$
$$f(x) = 4x\frac{d}{dx}(\sum_{n=0}^{\infty} x^n - 1)$$
$$f(x) = 4x\frac{d}{dx}(\frac{1}{1-x} - 1)$$
$$f(x) = 4x\frac{1}{(1-x)^2}$$

Hence $f(-1/4) = -1/(5/4)^2 = -16/25$.

4. (2 points) Suppose that you have an ice cream machine that produces ice cream at a rate of 10 gallons per day and that your friends consume ice cream at a rate which is not only proportional to the amount of ice cream present, but the time of week (your friends are hungriest late in the week). Suppose also that you lose ice cream due to melting, but that the more ice cream there is, the more slowly it melts. Write a differential equation to describe the amount of ice cream in your life as a function of time, *making sure* to define any quantities you use. Let $y(t)$ denote the amount of ice cream at time t. We have three things which may cause the ice cream amount to change: the ice cream machine making more, your friends eating it, and melting. The first is the simplest. The rate of ice cream production is given as 10 gallons per day. So

$$R_{\text{machine}} = 10$$

. The next is more complicated, because we have know that your friends are fickle and do not consume ice cream at a constant rate, but at a rate which depends on the amount of ice cream present as well as the time of week. So this rate looks something like

$$R_{\text{friends}} = -c(t)y$$

where c is a positive function of time. We are measuring time in days, so one form $c(t)$ could take is a periodic function which has period 7 days and which is smallest when $t = 0$. For example:

$$c(t) = K(1 - \cos(2\pi t/7))$$

for a positive constant K gives us the kind of behavior we want. Now, the last term needs to give us a negative rate that *decreases* in magnitude as y gets bigger. So this could be of the form

$$R_{\text{melt}} = -\frac{B}{1+y}$$

for some positive constant B. So our full equation reads

$$\frac{dy}{dt} = R_{\text{machine}} + R_{\text{friends}} + R_{\text{melt}}$$

which for the specific functions here is

$$\frac{dy}{dt} = 10 - K(1 - \cos(2\pi t/7))y + \frac{B}{1+y}$$

Remark 1: Note that this is just one possible form, many other (well-explained, well-thought out!) answers are possible.
Remark 2: This would be a very difficult equation to solve at is is nonlinear and nonautonomous (terms you will know soon!).

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