# MATH 1B: CALCULUS DISCUSSION SECTION 2: QUIZ 8

Please write neatly and show all your work.
**Useful info:** The binomial series is defined as follows:

$$(1+x)^k \sum_{n=0} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

and is convergent when $|x| < 1$.

1. (3 points) Find a the first 3 terms of a Taylor expansion about $a = 2$ for

$$f(x) = x^2 + 2x + 7$$

State the interval of convergence and estimate the error using Taylor's Theorem.

$$f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + R_3$$

$f(2) = 4+4+7 = 15$

$f'(2) = 4+2 = 6$

$f''(2) = 2$

$\Rightarrow f(x) = 15 + 6(x-2) + (x-2)^2 + R_3$

$|R_3| \leq \frac{M x^3}{3!}$ since $f'''(x) = 0$ for all $x$, $|R_3| \leq 0 \Rightarrow R_3 = 0$

(answer is EXACT). Finite series $\Rightarrow$ converges for all $x$.

2. (3 points) Use the first two terms of an infinite series expansion to estimate

$$\int_0^{.5} \frac{1}{(1+x^2)^{1/3}}\,dx$$

State and justify an upper bound on the error. CONTINUED ON OPPOSITE SIDE!

$$\frac{1}{(1+x^2)^{1/3}} = (1+x^2)^{-1/3} = 1 - \frac{1}{3}(x^2) + \frac{(-\frac{1}{3})(-\frac{4}{3})}{2!}(x^2)^2 + \frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{3!}(x^2)^3$$

$$= 1 - \frac{1}{3}x^2 + \frac{4}{18}x^4 - \frac{28}{81\cdot 3!}x^6 + \dots$$

$$\int_0^{0.5} \frac{1}{(1+x^2)^{1/3}}\,dx = \int_0^{0.5} 1 - \frac{1}{3}x^2 + \frac{4}{18}x^4 - \frac{28}{81\cdot 3!}x^6 + \dots\,dx$$

$$= \underbrace{x - \frac{1}{3\cdot 3}x^3 + \frac{4}{18\cdot 5}x^5 - \frac{28}{7\cdot 81\cdot 3!}x^7 + \dots}\Big|_0^{0.5}$$

$\nwarrow$ series is alternating, so |Error| $\leq$ |next term|

$$= 0.5 - \frac{1}{9}\left(\frac{1}{2}\right)^3 + \text{Error}$$

$$= \frac{1}{2} - \frac{1}{9\cdot 8} + \text{Error}$$

where $|\text{Error}| \leq \frac{4}{18\cdot 5}\left(\frac{1}{2}\right)^5$

Note: if we wanted to use Taylor's Theorem, we'd say $|E| \leq \frac{M x^5}{5!}$ where $M \geq |f^{(5)}(x)|$ $\leftarrow$ need to find $5^{th}$ derivative!!

3. (3 points) Use Taylor series to evaluate the limit

$$\lim_{x\to 0}\left(\frac{1-x^4-\cos(\sqrt{2}x^2)}{2x^8}\right)$$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

$\Rightarrow \frac{1-x^4-\cos(\sqrt{2}x^2)}{2x^8} = \frac{1-x^4-\left(1-\frac{(\sqrt{2}x^2)^2}{2!}+\frac{(\sqrt{2}x^2)^4}{4!}-\frac{(\sqrt{2}x^2)^6}{6!}+\dots\right)}{2x^8}$

$= \frac{-\frac{4x^8}{24}+\frac{2^3x^{12}}{6!}+\dots}{2x^8}$

$= \frac{1}{12} + \frac{8}{6!\cdot 2}x^4 + \dots$

so $\lim_{x\to 0}\left(\frac{1+x^4-\cos(\sqrt{2}x^2)}{2x^8}\right) = \lim_{x\to 0}\left(\frac{-1}{12}+\frac{8}{6!\cdot 2}x^4+\dots\right) = -\frac{1}{12}$ ✓

4. (1 point) Find an *equilibrium* solution for the differential equation

$$\frac{dy}{dx} = y^2(\ln(y+1))$$

equilibrium solution $y$: s.t. $\frac{dy}{dx} = 0$

Let $y=0$

$\frac{dy}{dx} = 0 = y^2\ln(y+1)$.

Prepared by ANNA LIEB. Comments and questions: **lieb@math.berkeley.edu**