MATH 128B: QUIZ 5

4 pts total

1. (Nonlinear BVP) Given the nonlinear problem on the interval $1 \le x \le 2$:

2 pts

$$y'' = -e^{-2y}, \quad y(1) = 0, y(2) = \ln(2)$$

Suppose I give you a value of $h = 1/N$. Let $x_i = 1 + ih$, $i = 0, \dots N+1$. Define an approximate solution $w_i \approx y(x_i)$ at each point $i = 0, \dots N+1$.

(a) Use a second-order finite difference approximation to represent the left-hand side and set up a system of equations of the form

$$F_k(w_1, \dots w_{N+1}) = 0$$

which will be satisfied by the $\{w_i\}, i = 1, \dots N+1$.

$$y''(x_i) \approx \frac{w_{i-1} - 2w_i + 2w_{i+1}}{h^2} \qquad i = 1, \dots N$$

$$\Rightarrow \quad \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} = -e^{-2w_i}$$

$$\Rightarrow \quad w_{i-1} - 2w_i + w_{i+1} + h^2 e^{-2w_i} = 0$$

Let $F_k(w_1, \dots w_{N+1}) = w_{k-1} - 2w_k + w_{k+1} + h^2 e^{-2w_k}$

Note $w_0 = y(1) = 0$, $w_{N+1} = y(2) = \ln(2)$ are set by B.Cs

(b) How many unknowns are there in this problem?

only N : we have values for $\begin{cases} w_0 = 0 \\ w_{N+1} = \ln(2) \end{cases}$

from boundary conditions

(c) Suppose we will solve the nonlinear system of equations above using Newton's Method. Indicate the sparsity pattern of the Jacobian matrix (where are the nonzero entries?).

solve $\underline{F}(\underline{x}) = 0$ using Newton $\Rightarrow$

$J_{ij} = \dfrac{\partial F_i}{\partial x_j}$ ... uses x_{j-1}, x_j and x_{j+1} $\Rightarrow$

$$J = \begin{pmatrix} \ddots & \ddots & 0 \\ \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots \end{pmatrix}$$

zero

potentially non zero

(tridiagonal)

CONTINUED ON OPPOSITE SIDE!

2. (Hat functions and Rayleigh Ritz)

2 pts

(a) Given the BVP:

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) = f(x), \quad y(0) = y(1) = 0$$

Show that

$$\int_0^1 p(x)\frac{dy}{dx}\frac{d\phi}{dx}dx = \int_0^1 f(x)\phi(x)dx$$

for any $\phi \in C_0^2([0,1])$.

multiply first eqn by ϕ and integrate by parts

$$\int_0^1 -\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)\phi(x)\,dx = \int_0^1 f(x)\phi(x)\,dx$$

$$\int_0^1 -\frac{d}{dx}\left(p(x)\frac{dy}{dx}\phi(x)\right) + p(x)\frac{dy}{dx}\frac{d\phi}{dx}\,dx = \int_0^1 f(x)\phi(x)\,dx$$

$$\cancel{-p(x)\frac{dy}{dx}\phi(x)\Big|_0^1}^{\,0} + \int_0^1 p(x)\frac{dy}{dx}\frac{d\phi}{dx}\,dx = \int_0^1 f(x)\phi(x)\,dx$$

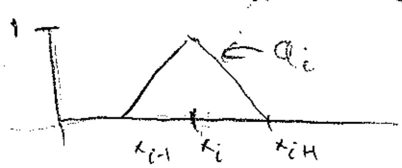
← since $\phi \in C_0^1[0,1] \Rightarrow \phi(0) = \phi(1) = 0$

(b) Take the interval $0 \le x \le 1$ with points $x_1 = 0, x_2 = 1/h, \ldots x_{N+1} = 1$. Show that the piecewise linear basis functions $\{\phi_i(x)\}, i = 2, \ldots N$ are independent. Draw a picture of a representative function.

suppose $\sum_i c_i\phi_i(x) \equiv 0$

Then for each j, $\sum_i c_i\phi_i(x_j) = 0$

since $\phi_i(x_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$



then this means $c_j\phi_j(x_j) = c_j = 0 \quad \forall j$.

3. (Bonus!) Who is your favorite fairy-tale character? I don't know if it counts... but the seagull from the Disney little mermaid is pretty fantastic!

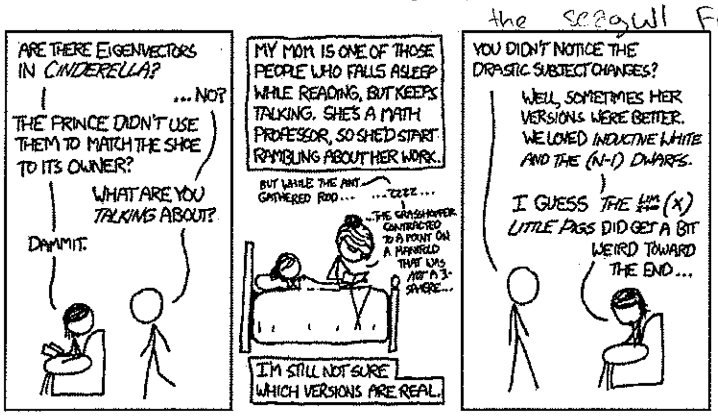


Figure 1: http://xkcd.com/872/

Prepared by ANNA LIEB. Comments and questions: **lieb@math.berkeley.edu**