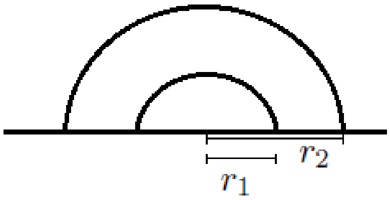
MATH 1B: CALCULUS DISCUSSION SECTION 2: QUIZ 4

Please write neatly and show all your work.

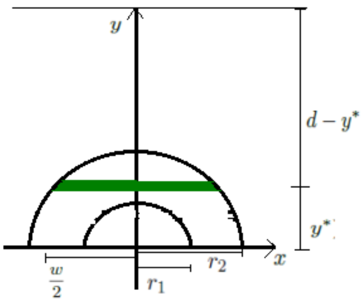
1. (2 points) Homer Simpson drops his doughnut (which has inner radius r_1 and outer radius r_2) in a lake of depth d and it lands exactly half-buried on the bottom as shown below. Homer is frantic to to know how much force is pressing on his precious donut.



(2 points) Set up (but do not evaluate) the expression you would need to calculate the hydrostatic force on the vertical face of the doughnut. Make sure you define any quantities which appear in your expression (it will help to draw a picture!). Useful facts: the pressure at depth x is equal to δx, where δ is a constant, pressure is the same in all directions, and pressure is equal to force divided by area.

Solution

Introduce the coordinate system in the figure below.



The equations of the inner and outer circles are given by $x^2 + y^2 = r_1^2$ and $x^2 + y^2 = r_2^2$, respectively. We can solve this problem by considering the force on the outer circle and then subtracting the force on the inner circle. For each circle, draw a thin horizontal slice whose center is at height y^* and whose width is Δy. Suppose that Δy is small so that the pressure is approximately constant. The pressure at depth y^* is given by

$$P = \Delta(d - y^*).$$

The area of the slice is given by $w\Delta y$, where w is the width of the slice. The equation for each circle tells us that $w = 2\sqrt{(}r_1^2 - (y^*)^2)$ for the inner circle and $w = 2\sqrt{(}r_2^2 - (y^*)^2)$ for the outer circle. The force on each slice is the pressure times the area. The force on the circle is then found by summing up over all these slices and then taking the limit as $\delta y \to 0$:

$$
\begin{aligned}
F_{outer} &= \int_0^{r_2} \delta(d-y)\sqrt{r_2^2-y^2}dy \\
F_{inner} &= \int_0^{r_1} \delta(d-y)\sqrt{r_1^2-y^2}dy \\
F &= F_{outer} - F_{inner} \\
F &\int_0^{r_2} \delta(d-y)\sqrt{r_2^2-y^2}dy - \int_0^{r_1} \delta(d-y)\sqrt{r_1^2-y^2}dy
\end{aligned}
$$

2. (6 points) Determine if the following sequences are convergent or not and explain your reasoning clearly. You will receive one point for the correct answer, and one point for your justification.

(a)

$$a_n = \sin\left(\frac{2n\pi + \sqrt{7}}{2n}\right)$$

Solution The sequence is convergent. If we write $a_n = \sin(b_n)$, then we have

$$\lim_{n\to\infty} b_n = \pi$$

and since $\sin(x)$ is continuous at π,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin(b_n) = sin(\lim_{n\to\infty} b_n) = \sin(\pi) = 0.$$

(b)

$$b_n = -\ln\left(\frac{1}{n+1}\right)$$

Solution The sequence diverges to ∞. We have a theorem that if $f(x)$ is a function such that $f(n) = b_n$ for all n, then

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x).$$

Let $f(x) = \ln\left(\frac{1}{x+1}\right) = -\ln(x+1)$. Since

$$\lim_{x\to\infty} f(x) = -\infty,$$

the sequence diverges.

(c)

$$c_n = \left(\frac{3}{5}\right)^n (-1)^n n!$$

Solution The sequence diverges. To see this, write

$$
\begin{aligned}
|c_n| &= \left(\frac{3}{5}\right)\left(2\frac{3}{5}\right)\cdots\left(n\frac{3}{5}\right) \\
|c_n| &\geq \left(\frac{3}{5}\right)^2 n
\end{aligned}
$$

So the sequence $|c_n|$ diverges to ∞ by comparison with the sequence n. Hence the sequence c_n also diverges, because it is unbounded.

3. (2 points) Explain why the sequence defined recursively by

$$a_1 = 3$$
$$a_{n+1} = \frac{1}{4 - a_n}$$

converges and find its limit.

Solution The sequence has a limit if it is monotone and bounded. First we show that it is monotone. To do this, we will use mathematical induction.

We first show that if $a_n \leq a_{n-1}$, then $a_{n+1} \leq a_n$. Suppose $a_n \leq a_{n-1}$. Then

$$4 - a_{n-1} \leq 4 - a_n$$
$$\frac{1}{4 - a_n} \leq \frac{1}{4 - a_{n-1}}$$
$$a_{n+1} \leq a_n$$

Now, since $a_1 = 3$ and $a_2 = 1$, we have that the property $a_n \leq a_{n-1}$ holds for $n = 2$. But we just showed that if it holds for $n = 2$, then it must hold for $n = 3$. But if it holds for $n = 3$, it must hold for $n = 4$, etc. So it holds for all n and a_n is monotone decreasing. Specifically, $a_n < 4$ for all n, so we can bound a_n from both above and below:

$$0 \leq a_n < 4$$

Hence a_n is monotone and bounded and the sequence converges.

Now, to find the limit, we note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$; call this limit L. Then

$$L = \frac{1}{4 - L}$$

Solving the quadratic equation for L we get that

$$L = 2 \pm \sqrt{3}$$

But since a_n is monotone decreasing and $2 + \sqrt{(3)} > 3 > a_2$, the negative solution is the correct one.

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