MATH 128B: QUIZ 2 SOLUTIONS

Useful facts:
$\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right)$
$2^{15} \approx 32000$

1. (Chebyshev Polynomias) Use the definition of the n^{th} Chebyshev polynomial $T_n(x) = \cos(n \arccos(x))$ to find an expression for the product $T_i(x)T_j(x)$ which has no products of Chebyshev polynomials. **Solution** Using the trig formula we find

$$T_i(x)T_j(x) = \cos(i \arccos(x))\cos(j \arccos(x)) = \frac{1}{2}\left[\cos((i+j)\arccos(x)) + \cos((i-j)\arccos(x))\right]$$

Since $\cos(x)$ is an even function we have

$$T_i(x)T_j(x) = \frac{1}{2}\left(T_{i+j}(x) + T_{|i-j|}(x)\right)$$

2. (Fourier Series)

(a) Find the coefficients in the general continuous least-squares polynomial $S_n(x)$ defined as

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)$$

$$f(x) = \begin{cases} 1, & \text{if } -\pi < x \leq 0 \\ 0 & \text{if } 0 < x \leq \pi \end{cases}$$

The coefficients are given by

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(kx) f(x)dx, \quad k = 0, 1, \ldots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(kx) f(x)dx, \quad k = 1, 2 \ldots$$

Plugging in, we find

$$a_0 = 1, \quad a_k = \frac{1}{\pi}\int_{-\pi}^{0} \cos(kx)dx = \frac{1}{k\pi}\sin(k\pi)|_{-\pi}^{0} = 0, \quad k \geq 1$$

$$b_k = \int_{-\pi}^{0} \sin(k\pi)dx = -\frac{1}{k\pi}\cos(k\pi)|_{-\pi}^{0} = \begin{cases} -\frac{2}{k\pi} & k \text{ odd} \\ 0 & \text{even} \end{cases}$$

CONTINUED ON OTHER SIDE

(b) Suppose you want to compute the Fourier coefficients for data on a grid with thirty two thousand (3.2×10^4) points. Estimate the *order of magnitude* (yes, you can do this without a calculator) of the number of calculations you must perform to do this
(i) with a direct method
(ii) via FFT.

(i) $\mathcal{O}(2n)^2 \approx (6 \times 10^4)^2 = 36 \times 10^8 = 3.6 \times 10^9$: order 10^9 (1 billion)
(iI) $\mathcal{O}(n \log(n)) \approx 3 \times 10^4 \times 15 = 4.5 \times 10^5$ order: 10^5 (ten thousand times less effort!).

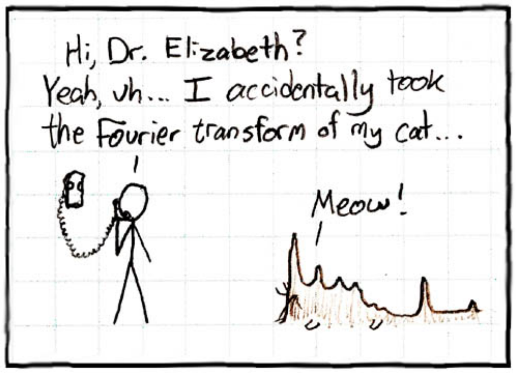

Figure 1: http://xkcd.com/26/

3. Bonus: Draw an FFT of yourself.

Prepared by ANNA LIEB. Comments and questions: **lieb@math.berkeley.edu**