MATH 1B: CALCULUS DISCUSSION SECTION 2 QUIZ 10 SOLUTIONS

1. Find a general solution for

   (a) (2 points) $y' + y = \cos(e^x)$
   Use integrating factor $I(x) = e^x$. Then

   $$\frac{d}{dx}\left(e^x y\right) = e^x \cos(e^x)$$

   So we integrate:

   $$e^x y(x) = \int e^x \cos(e^x) dx$$

   using the substitution $u = e^x$ we find

   $$e^x y(x) = \sin(e^x) + c$$
   $$y(x) = e^{-x}\sin(e^x) + ce^{-x}$$

   (b) (2 points) $y'' = 3y'$
   This is a second order homogenous linear constant coefficient equation, so try a solution of the form $y = e^{rx}$. The equation r must satisfy is $r^2 - 3r = 0$, so the solutions are $r = 0, r = 3$. The general solution is

   $$y(x) = c_1 + c_2 e^{3x}$$

   (c) (2 points) $2y'' + 2y' + y = 0$
   As above, we seek a solution of the form e^{rx}. So r must satisfy $2r^2 + 2r + 1 = 0$ Which we solve using the quadradtic formula to find that $r = 1 \pm 2i$ so the general solution is

   $$y(x) = e^{-x/2}\left(c_1 \cos(x) + c_2 \sin(x)\right)$$

2. (2 points) Determine a solution to the boundary value problem

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0, \quad y(0) = 1, \quad y(1) = 0$$

or explain why such a solution does not exist.
The characteristic equation has a double root at $r = 2$. Hence the general solution is

$$y(x) = c_1 e^{2x} + c_2 x e^{2x}$$

At $x = 0$, we apply the boundary condition to find that $y(1) = c_1 = 1$. At $x = 1$, we apply the boundary condition to find that $e^2 + c_2 e^2 = 0$ and thus $c_2 = -1$. Therefore the solution to the boundary value problem is

$$y(x) = e^{2x}\,(1 - x)$$

3. A tank contains 1000 L of water. Suppose that a grape Kool-Aid solution at a concentration of 100 grams per L is being piped into the tank at a rate of 10 L per minute. Suppose you are draining the tank at a rate of 20 L per minute. Assume that the Cool-Aid stays very well mixed.

   (a) (1 point) Let $y(t)$ be the amount of Kool-Aid (in grams) in the tank at time t. Write down a differential equation for how this amount changes as a function of time.

   We write $\frac{dy}{dt} = R_{in} - R_{out}$. R_in is given by the volume of liquid entering the tank multiplied by the concentration of Cool-Aid in this liquid. Thus

$$R_{in} = \left(10\frac{\text{L}}{\text{min}}\right)\left(100\frac{\text{g}}{\text{L}}\right)$$

   (Notice thate R_{in} has units of g/min, as we expect. R_{out} is a little trickier. We are assuming that the tank is well mixed, so at any given point in time, the concentration of Cool-Aid in the liquid is given by the average; that is, the concentration is equal to $y(t)$/(Volume of tank). The volume of the tank is not a constant; it changes because the liquid leaves the tank more quickly than it enters. We can figure out an expression for $v(t)$, the volume of the tank at time t, by solving the simple differential equation $dv/dt = -10$, with initial condition $v(0) = 1000$. So $v(t) = 1000 - 10t$, and the concentration in the tank at time t is given by $y/(1000 - 10t)$. Hence we have

$$R_{out} = \left(20\frac{\text{L}}{\text{min}}\right)\left(\frac{y}{1000 - 10t}\frac{\text{g}}{L}\right)$$

   Thus the differential equation for y is

$$\frac{dy}{dt} = 1000 - \frac{2y}{100 - t}$$

   (b) (1 point) Solve the differential equation.
   Rewrite the equation as

$$y' + \frac{2}{100 - t}y = 1000$$

The equation is not seperable, so we use an integrating factor

$$I(t) = e^{\int \frac{2}{100-t}\,dt} = e^{-2\log(100-t)} = (100-t)^{-2}$$

Then we have

$$\frac{d}{dt}\left((100-t)^{-2}y\right) = 1000(100-t)^{-2}$$

Integrating and solving for y, we find that

$$y = 1000(100-t) + C(100-t)^2$$

With initial condition $y(0) = 0$, the solution is

$$y(t) = 1000(100-t) + 10(100-t)^2$$