THINKING ABOUT SEQUENCES AND LIMITS

A sequence of numbers is a list of numbers with a definite order. Each element has an *index*, n, which tells you where on the list it is. Given a sequence of terms $\{a_n\}$, where n is the index, you might want to know what happens as you go very far down along the list. Can you say what the n^{th} entry on the list will be as you make n infinitely large? If you can, and if that entry is finite, then we say the sequence is *convergent*. Mathematically, we say that a sequence $\{a_n\}$ converges to a limit, L, in the following way:

$$\lim_{n\to\infty} a_n = L$$

when for any $\epsilon > 0$, there exists a number N such that $|a_n - L| < \epsilon$ whenever $n > N$.

What does this definition mean? It means that as you go out far enough in your list of numbers, they get closer and closer the limit. More precisely, if the limit is L, you must show that for any possible $\epsilon > 0$ (no matter how small), it is possible to find a point in the list of terms such that all terms farther down the list are within distance ϵ of L. That "point" on the list is the number N. The limit def intion says that given any positive ϵ, we need to find an N such that if we look at any a_n where the index n is larger than N, we know that $|a_n - L| < \epsilon$. What if the limit does NOT exist? We must show that there is some ϵ such that no matter how large we make N, we can still find a term on the list a_n such that $|a_n - L| \geq \epsilon$. Let's do some examples:

Example 1 Consider the sequence defined by

$$a_n = \frac{1}{n}.$$

We will show that

$$\lim_{n\to\infty} (a_n) = 0.$$

To do this, we must, for ANY ϵ, be able to find an N large enough that $|a_n - 0| < \epsilon$ whenever $n > N$. Let N be the smallest integer such that $N > 1/\epsilon$. Then

$$|a_n - 0| < 1/N < \epsilon$$

for all $n > N$, so the limit exists.

Example 2 Consider the sequence

$$\{c_n\} = \{0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots\}$$

Let's prove that this sequence *does not* converge to 0. To do this, let $\epsilon = 1/2$. Now suppose you try to find N such that $|c_n| < 1/2$ for all $n > N$. But this can't be true, because no matter how large we make N, we can keep going down the list and eventually find another term of the sequence where $c_n = 1 > 1/2$. So the limit cannot be 0. Similarly, you can show the limit of this sequence cannot be 1, so the sequence does not converge at all.

How can we tell, given a list of numbers, if it is convergent or not? Consider the following sequences of numbers:

n	a_n	b_n
1	2.35	-9.00
2	3.07	31.50
3	3.70	-90.00
4	4.254	184.375
5	4.77	-307.70
$\vdots$	$\vdots$	$\vdots$
10	6.5550	549.04

Which of these two sequences seems convergent? These numbers came from the following sequences of partial sums:

$$a_n = \sum_{j=2}^{n+2} \frac{1}{\ln(j)}$$

$$b_n = \sum_{j=0}^{n} \frac{1}{j!}(-9)^j$$

You can show that a_n never stops growing, albeit slowly; the series

$$\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$$

is divergent. But, in fact, the series

$$\sum_{n=0}^{\infty} \frac{1}{n!}(-9)^n$$

actually converges to the value e^{-9}. Thus the behavior of any finite number of terms does not guarantee anything about the behavior of the infinite sequence. In some cases looking at the first few terms can help give you an idea of what you might like to prove; but as this example shows, they may also mislead you!

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