Householder's Method

Idea: We have an arbitrary symmetric matrix A. A would be a lot nicer to work with if it were TRIDIAGONAL (for example, we have marvelous ways of inverting, or finding eigenvalues for, a TRIDIAGONAL matrix.)

But A isn't tridiagonal. Bummer...?

Householder's method is a way of finding a tridiagonal matrix $\tilde{A}$, which is SIMILAR to A.

Here's what A looks like:

$$\begin{pmatrix} x & x & x & \cdots & x \\ x & x & x & \cdots & \vdots \\ x & & \ddots & & \\ x & & & \ddots & \\ x & x & \cdots & & x \end{pmatrix}$$

everything is potentially non-zero!

Here's what $\tilde{A}$ looks like:

$$\begin{pmatrix} x & x & 0 & 0 & \cdots & 0 \\ x & x & x & 0 & \cdots & 0 \\ 0 & x & x & x & \ddots & \vdots \\ 0 & 0 & x & \ddots & \ddots & 0 \\ \vdots & & & \ddots & x & x \\ 0 & 0 & 0 & \cdots & x & x \end{pmatrix}$$

only non-zero entries are on main diagonal, or immediately next to it!

To find $\tilde{A}$, we will apply a series of <u>Householder</u> <u>Transformations</u> to A.

* Think of these as successively zeroing out the bits we don't like*

Consider the first coulmn of A: $a_{:,1} = \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{pmatrix}$

we can alter it by acting some matrix Q_1^T on it:

↑ transpose so notation is consistent w/ slides

$$Q_1^T a_{:,1} \Rightarrow \tilde{a}_{:,1}$$

we want $\tilde{a}_{i,1}$ to have this form:

$$a_{i,1} = \begin{pmatrix} a_{11} \\ \text{stuff} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \} \text{ unchanged} \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \text{a multiple of } e'_{n-1\times 1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{matrix}$$

so Q_1^T has this form:

want identity here so we don't touch a_{11}!

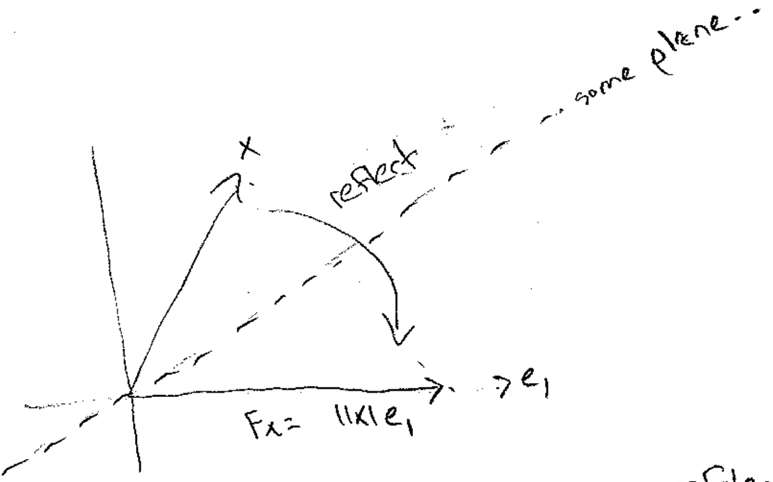
$$\begin{pmatrix} I_{1\times 1} & 0 \cdots 0 \\ 0 & \\ \vdots & \underbrace{F_{n-1}}_{n-1} \\ 0 & \end{pmatrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n-1$$

F_{n-1} acts on the last $n-1$ components of $a_{:,1}$

Let $x = \begin{pmatrix} a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$ (so $a_{:,1} = \begin{pmatrix} a_{11} \\ | \\ x \\ | \end{pmatrix}$

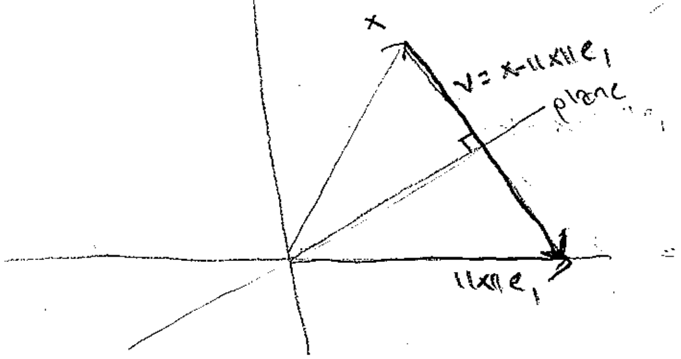
want: $Fx = \begin{pmatrix} \|x\| \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

graphically:



we can think of the transformation F as reflecting x across some plane so that the end result is a multiple of $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Since the length is preserved, the result will be $\begin{pmatrix} \|x\|_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \|x\|_2 e_1$

This kind of reflection has a name- its a **Householder Reflection**.



the plane we want is orthogonal to $v = \|x\|_2 e_1 - x$

So to construct the right reflection, we need to flip the sign on anything in the v direction.

$$x \nearrow = \underbrace{\nwarrow av}_{\text{some multiple of } v} + \underbrace{w \nearrow}_{\text{some combination of stuff } \perp \text{ to } v} \qquad \begin{array}{c} w = a_1 w_1 + \dots a_n w_k \\ w_i^T v = 0 \end{array}$$

we want $x \to \underset{\substack{\uparrow \\ \text{sign flips}}}{-av} + \underset{\substack{\uparrow \\ \text{stays the same}}}{w}$

The matrix

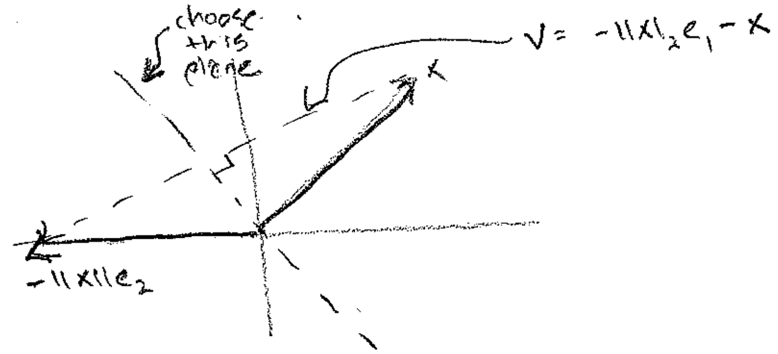
$$F = I - 2\frac{vv^T}{v^Tv} \quad \text{achieves this.}$$

$$Fx = \left(I - \frac{2vv^T}{v^Tv}\right)(av + a_1 w_1 + \dots a_k w_k)$$

$$= av + a_1 w_1 + \dots a_k w_k - 2a\frac{vv^Tv}{v^Tv} - \left(a_1 v \frac{\overset{0}{\cancel{v^T w_1}}}{v^Tv} + \dots a_k v \frac{\overset{0}{\cancel{v^T w_k}}}{v^Tv}\right)$$

$$Fx = -av + w \quad \checkmark$$

Note we could also have reflected $\|x\|$ on to $-\|x\|_2 e_1$



It turns out the method is more numerically stable if we choose

$$v = \begin{cases} -\|x\|_2 e_1 + x & \text{if } x_1 < 0 \\ \|x\|_2 e_1 + x & \text{if } x_1 \geq 0 \end{cases}$$

so we sum this up as

$$v = \operatorname{sign}(x_1)\|x\|_2 e_1 + x$$

So this gives us what we need to deal with the first column. To deal with the first row, we can think of applying Q^1 to A^T ... and then transposing back

$$Q_1^T A \to \begin{pmatrix} x & x & & \\ x & x & x & \\ 0 & & x & \\ 0 & x & & x \\ 0 & & & \end{pmatrix}$$

$$Q_1^T A^T \to \begin{pmatrix} x & x & \vdots \\ x & x & \\ 0 & x & \\ \vdots & x & \\ 0 & \cdot & \cdot \end{pmatrix}$$

zeroed out the first row entries we didn't want!

$$(Q_1^T A^T)^T = \begin{pmatrix} x & x & 0 & \cdots & 0 \\ x & x & & & \\ x & x & & & \\ x & & & & \\ x & & & & \end{pmatrix} = A^{TT} Q_1^{TT} = A Q_1$$

So in total, if we take

$$Q_1^T A Q_1$$

we get

$$Q_1^T A Q_1 = \begin{pmatrix} x & x & 0 & \cdots & 0 \\ x & x & x & x & x \\ 0 & x & & & \\ \vdots & x & \ddots & & \\ 0 & x & & & \end{pmatrix}$$

Now we do it again, only we want

$$Q_2^T = \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & \underbrace{F_{n-2}}_{n-2} \end{pmatrix} \}\, n-2$$

where F_{n-2} acts on the last $\underline{n-2}$ entries A of the <u>second</u> column of A

$$F_{n-2} \begin{pmatrix} a_{32} \\ a_{42} \\ \vdots \\ a_{n2} \end{pmatrix} = \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Rightarrow$ F_{n-2} is a Householder reflection made with

$x = \begin{pmatrix} a_{32} \\ \vdots \\ a_{n2} \end{pmatrix}$ and $v = \text{sign}(x)\|x\|_2 \underset{\uparrow}{e_1} + x$
now an $n-2 \times 1$ vector!

... and so on for $Q_3, Q_4 \ldots Q_{n-2}$.

Example (textbook # 1c)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$ $3\times 3 \Rightarrow$ only need to do 1 step!

$$Q_1^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{F_2} \\ 0 & & \end{pmatrix}$$

F_2 is a householder reflection formed with $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

so $v = (+1)(\sqrt{2})\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}+1 \\ 1 \end{pmatrix}$ $\quad \|v\| = v^Tv = (\sqrt{2}+1)^2 + 1 = 2(2+\sqrt{2})$

And $F_2 = I - 2\frac{vv^T}{v^Tv} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{2(2+\sqrt{2})}\begin{pmatrix} (\sqrt{2}+1)^2 & \sqrt{2}+1 \\ \sqrt{2}+1 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

(algebra ... verify on your own. $Q_1^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

so $$Q_1^T A Q_1 = \begin{pmatrix} 1 & -\sqrt{2} & 0 \\ -\sqrt{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$