# Rational Function Approximation with Chebyshev Polynomials

$$f(x) \approx r(x) = \frac{\sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)} \qquad \text{with } q_0 = 1 \quad \text{and} \quad n + m = N$$

write $f(x)$ as a series of Chebyshev Polynomials:

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x) \qquad (a_k \text{ known})$$

we will solve for the unknown coefficients $\{p_k\}_{k=0}^{n}$ and $\{q_k\}_{k=1}^{m}$ by asserting that

$$f(x) - r(x) = \sum_{k=0}^{\infty} a_k T_k(x) - \frac{\sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)}$$

has no terms of degree $\leq N$.
(from here, drop $x$-dependence and write $T_k$ instead of $T_k(x)$, for brevity)

$$f(x) - r(x) = \frac{\sum_{j=0}^{m} \sum_{i=0}^{\infty} a_i q_j T_i(x) T_j(x) - \sum_{l=0}^{n} p_l T_l(x)}{\sum_{k=0}^{m} q_k T_k(x)}$$

(put over common denominator)
(and relabel sums to avoid confusion)

so we want $w(x) = \sum_{j=0}^{m} \sum_{i=0}^{\infty} a_i q_j T_i T_j - \sum_{l=0}^{n} p_l T_l$ to have no $T_k$ for $k \leq N$.

$\boxed{\text{Goal:}}$ write $w(x) = \sum_{l=0}^{\infty} (\text{stuff})_l T_l$.

If we can do that, use linear independence of $\{T_l\}$ to assert $(\text{stuff})_l = 0$ for each $l = 0, 1, \ldots, N$

$\boxed{\text{Step 1:}}$ expand product $T_i T_j$ using handy identity

$$T_i T_j = \frac{1}{2}\left[T_{i+j} + T_{|i-j|}\right]$$

Note: this tells us we dont actually have to take the sums $\sum_{k=0}^{\infty} a_k T_k$ ← don't need to worry about all $a_k$. We only worry about $T_l$ if $l \leq n$

$\Rightarrow$ so we can write $w(x)$ as

$$w(x) = \sum_{j=0}^{m} \sum_{i=0}^{N+m} a_i q_j T_i T_j - \sum_{l=0}^{n} p_l T_l$$

← only go up to $N+m$

(really there's $O(T_{N+1})$ here) since we don't care about terms with $T_l$ for $l > N$.

so we have

$$w(x) = \sum_{j=0}^{m} \sum_{i=0}^{N+m} \frac{a_i q_j}{2}\left(T_{i+j} + T_{|i-j|}\right) - \sum_{l=0}^{n} q_l T_l$$

$\boxed{\text{Step 2: reindex}}$

$$w(x) = \sum_{j=0}^{m} \Bigg[ \underbrace{\sum_{i=0}^{N+m} \frac{a_i q_j}{2} T_{i+j}}_{\substack{\text{reindex with} \\ l = i+j}} + \underbrace{\sum_{i=0}^{j} \frac{a_i q_j}{2} T_{j-i}}_{\substack{\text{reindex with} \\ l = j-i}} + \underbrace{\sum_{i=j+1}^{N+m} \frac{a_i q_j}{2} T_{i-j}}_{\substack{\text{reindex with} \\ l = i-j}} \Bigg] - \sum_{l=0}^{n} q_l T_l \quad \text{(expand } |i-j| \text{ into 2 sums)}$$

*poof* (reindexing magic)

$$w(x) = \sum_{j=0}^{m} \left[ \sum_{l=j}^{N+m+j} \frac{a_{l-j} q_j}{2} T_l + \sum_{l=j}^{0} \frac{a_{j-l} q_j}{2} T_l + \sum_{l=1}^{N+m-j} \frac{a_{l+j} q_j}{2} T_l \right] - \sum_{l=0}^{n} p_l T_l$$

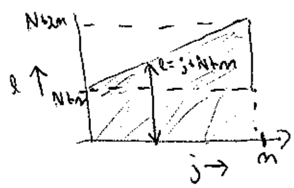
combine these - theses two terms with $a_j q_j T_0$ - so add one to this

$$w(x) = \sum_{j=0}^{m} \left[ \sum_{l=0}^{N+m+j} \frac{a_{|l-j|} q_j}{2} T_l + \sum_{l=0}^{N+m-j} \frac{a_{l+j} q_j}{2} T_l \right] - \sum_{l=0}^{n} p_l T_l$$

It's looking better! but not what we want yet. We've got more reindexing shenanigans in our future.

We're going to use the following: (look at picture to see why)

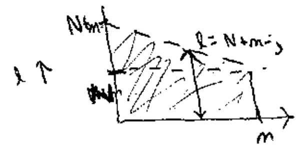
$$\sum_{j=0}^{m} \sum_{l=0}^{N+m+j} = \sum_{l=0}^{N+m} \sum_{j=0}^{m} + \sum_{l=N+m+1}^{N+2m} \sum_{j=N+2m-l}^{m}$$




(it's the discrete equivalent of exchanging order of integration in some 2D region)

and

$$\sum_{j=0}^{m} \sum_{l=0}^{N+m-j} = \sum_{l=0}^{N+m-n} \sum_{j=0}^{m} + \sum_{l=N+1}^{N+m} \sum_{j=0}^{N+m-l}$$




$\boxed{\text{step 3: use linear independence}}$

(why are we doing this?! Recall we need to pull out $T_l$)

So:

$$w(x) = \sum_{l=0}^{N+m} \sum_{j=0}^{m} \frac{a_{|l-j|} q_j}{2} T_l + \underbrace{\sum_{l=N+m+1}^{N+2m} \sum_{j=N+2m-l}^{m} \frac{a_{|l-j|} q_j}{2} T_l}_{\substack{l>N \text{ for every term...} \\ \text{DON'T need this sum!} \\ \text{(for what follows)}}} + \sum_{l=0}^{N} \sum_{j=0}^{m} \frac{a_{l+j} q_j}{2} T_l + \sum_{l=N+1}^{N+m} \sum_{j=0}^{N+m-l} \frac{a_{l+j} q_j}{2} T_l - \sum_{l=0}^{n} p_l T_l$$

keep terms $\leq N \Rightarrow$ assert $w(x) = 0 + O(T_{N+1})$ recall that $n+m=N$; clean up

$$\sum_{l=0}^{n} \left( \sum_{j=0}^{m} \frac{(a_{|l-j|} + a_{l+j})}{2} q_j - p_l \right) T_l + \sum_{l=n+1}^{n+m} \left( \sum_{j=0}^{m} \frac{(a_{|l-j|} + a_{l+j})}{2} q_j \right) T_l = 0$$

linear indedendence of $\{T_l\} \Rightarrow$

$$\sum_{l=0}^{n+m} c_l T_l = 0 \Rightarrow c_l = 0 \quad \left(\text{where } c_l = \begin{cases} \sum_{j=0}^{m} \frac{(a_{|l-j|} + a_{l+j})}{2} q_j - p_l & l \leq n \\ \sum_{j=0}^{m} \frac{(a_{|l-j|} + a_{l+j})}{2} q_j & n < l \leq n+m \end{cases}\right.$$

**Step 4:** write it out as a linear system!

we haven't used an important piece of information: $q_0 = 1$

so our equations read

$$c_\ell = 0 = \sum_{j=1}^{m} \left(\frac{a_{|\ell-j|} + a_{\ell+j}}{2}\right) q_j - p_\ell + a_\ell \overset{1}{\overset{\shortparallel}{q_0}} = 0 \qquad (\ell \le n)$$

↑ take out $j=0$ term

(similarly for $\ell > n$)

$$\text{so:} \quad \boxed{\begin{array}{ll} \displaystyle\sum_{j=1}^{m} \left(\frac{a_{|\ell-j|} + a_{\ell+j}}{2}\right) q_j - p_\ell = -a_\ell & 0 \le \ell \le n \\ \displaystyle\sum_{j=1}^{m} \left(\frac{a_{|\ell-j|} + a_{\ell+j}}{2}\right) q_j = -a_\ell & n < \ell < n+m \end{array}}$$

Linear system of equations! Yay! we know how to solve!

Make it pretty: Look at $\displaystyle\sum_{j=1}^{m} \left(\frac{a_{|\ell-j|} + a_{\ell+j}}{2}\right) q_j$ terms:

$$\ell = 0: \quad \sum_{j=1}^{m} \left(\frac{a_{|\ell-j|} + a_{\ell+j}}{2}\right) q_j = \sum_{j=1}^{m} a_j q_j = a_1 q_1 + a_2 q_2 + \dots a_m q_m$$

$$\ell = 1: \quad \begin{cases} \displaystyle\sum_{j=1}^{m} \frac{a_{|\ell-j|}}{2} q_j = \tfrac{1}{2}(a_0 q_1 + a_1 q_2 + a_2 q_3 + \dots a_{m-1} q_m) \\ \displaystyle\sum_{j=1}^{m} \frac{a_{\ell+j}}{2} q_j = \tfrac{1}{2}(a_2 q_1 + a_3 q_2 + \dots a_{m+1} q_m) \end{cases}$$

$$\ell = 2 \quad \begin{cases} \displaystyle\sum_{j=1}^{m} \frac{a_{|\ell-j|}}{2} q_j = \tfrac{1}{2}(a_1 q_1 + a_0 q_1 + a_1 q_3 + a_2 q_4 + \dots a_{m-2} q_m) \\ \displaystyle\sum_{j=1}^{m} \frac{a_{\ell+j}}{2} q_j = \tfrac{1}{2}(a_3 q_1 + a_4 q_1 + \dots + a_{m+2} q_m) \end{cases}$$

see a pattern? ~~for any~~ we have these systems of equations (for $\ell > 1$)

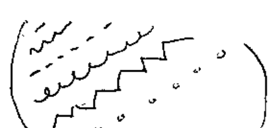
$$\underbrace{\frac{1}{2}\begin{pmatrix} a_0 & a_1 & \dots & a_{m-1} \\ a_1 & a_0 & a_1 \dots & a_{m-2} \\ a_2 & \ddots & & \\ a_{n+m-1} & a_n & \dots & \end{pmatrix}}_{\text{pertains to } a_{|\ell-j|} \text{ terms}} \begin{pmatrix} q_1 \\ \vdots \\ q_m \end{pmatrix} + \underbrace{\frac{1}{2}\begin{pmatrix} a_2 & a_3 & \dots & a_{m+1} \\ a_3 & a_4 & \dots & a_{m+2} \\ a_4 & & & \\ a_{n+m+1} & & \dots & a_{n+2m+1} \end{pmatrix}}_{\text{pertains to } a_{\ell+j} \text{ terms}} \begin{pmatrix} q_1 \\ \vdots \\ q_m \end{pmatrix}$$

this matrix is a TOEPLITZ MATRIX

↑ constant on diags

$$\overbrace{\begin{pmatrix} a_0 & a_1 & \dots & a_{m-1} \\ a_1 & a_0 & a_1 & \dots \\ a_2 & a_1 & a_0 & \\ a_n & & & \\ a_{n+m-1} & a_{n+m-2} & \dots & a_n \end{pmatrix}}^{n+m} \Bigg\} n+m$$

this matrix is a HANKEL MATRIX



↑ constant on antidiags

$$n+m \Bigg\{ \underbrace{\begin{pmatrix} a_2 & a_3 & a_4 & a_5 & \dots \\ a_3 & a_4 & a_5 & & \\ a_4 & a_5 & & & \\ a_5 & a_6 & & & \end{pmatrix}}_{n+m}$$

## Step 5: put it all together (whew, almost there!)

$$\underline{x} = \underbrace{(q_1 \ldots q_m, p_0, \ldots p_n)^T}_{m+n+1 \text{ unknowns}} \qquad \underline{a} = \underbrace{(-a_0, \ldots -a_{n+m})^T}_{n+m+1}$$

(known) ... use recurrence relations and integration to compute!

$$A = \begin{array}{c} \\ n+m\left\{ \vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}} \right. \end{array} \left( \underbrace{\begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ & \text{Toeplitz + Hankel here} & & \\ & & & \end{pmatrix}}_{m} \begin{matrix} \overbrace{\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & & O \\ O & & \ddots & -1 \end{pmatrix}}^{n} \\ O \end{matrix} \right) \begin{array}{c} \}n \\ \\ \end{array}$$

$A$ is $(n+m+1) \times (n+m+1)$

$$A\underline{x} = \underline{a}$$

so go type in Matlab "$\underline{x} = A \backslash \underline{a}$"

voila! you're done ☺