Solutions

MATH 128B: QUIZ 1

1. (Linear Systems)

(1 pt) (a) For the system of equations:

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

do one update step of Gauss-Seidel with initial guess $(1,1)^T$.

G-S update: $x_0 = \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad x_1 = \begin{pmatrix} x_1^1 \\ x_1^2 \end{pmatrix}$ $\quad x_1^1 = \frac{1}{a_{11}}(b_1 - a_{12}x_0^2) = \frac{1}{2}(2-1) = \frac{1}{2}$

$x_2^1 = \frac{1}{a_{22}}(b_2 - a_{21}x_1^1) = \frac{1}{3}(1 - 1(\frac{1}{2})) = \frac{1}{6}$

note! use updated value x_1^1 (Jacobi would use x_0^1)

$\Rightarrow \boxed{x^1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{6} \end{pmatrix}}$

(2 pt) (b) Consider solving the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with CG. Given a search direction p_{i-1}, the update step is

$$\mathbf{x}_i = \mathbf{x}_{i-1} + \alpha_i \mathbf{p}_{i-1}.$$

Explain how to choose α_i and derive a formula in terms of $\mathbf{A}$, $\mathbf{p}_{i-1}$ and the residual $\mathbf{r}_{i-1}$.

Choose α_i to minimize the function

$\Pi(x_i) = \frac{1}{2}x_i^T A x_i - b^T x_i = \frac{1}{2}(x_{i-1} + \alpha_i p_{i-1})^T A (x_{i-1} + \alpha_i p_{i-1}) - b^T(x_{i-1} + \alpha_i p_{i-1})$ w.r.t. α_i

$\Pi(x_i) = \frac{1}{2}(x_{i-1}^T A x_{i-1} + 2\alpha_i p_{i-1}^T A x_{i-1} + \alpha_i^2 p_{i-1}^T A p_{i-1}) - b^T x_{i-1} + \alpha_i b^T p_{i-1}$

Set $0 = \frac{\partial \Pi}{\partial \alpha_i} = p_{i-1}^T A x_{i-1} + \alpha_i p_{i-1}^T A p_{i-1} - b^T p_{i-1} \Rightarrow \alpha_i = \frac{(b^T - (Ax_i)^T)p_{i-1}}{p_{i-1}^T A p_{i-1}} = \boxed{\frac{r_i^T p_{i-1}}{p_{i-1}^T A p_{i-1}}}$

2. (Approximation)

(1 pt) (a) For each of the following functions, decide if it can serve as a weight function on the interval (0,1):

(i) $w_1(x) = \frac{1}{\sqrt{1-x^2}}$ yes

(ii) $w_2(x) = \sin(x)$ yes

(iii) $w_3(x) = e^{-x^2}$ yes

actually all of these functions are ≥ 0 on $(0,1)$ and are not $\equiv 0$ on any subinterval. But the $\sin(x)$ one is tricky – on e.g. $(0, 2\pi)$, $\sin(x)$ is NOT a valid weight function

(1 pt) (b) State what it means for the functions $\{\phi_0, \phi_1, \ldots, \phi_n\}$ to be linearly independent.

$\{\varphi_0, \ldots \varphi_n\}$ lin. independent if whenever we have constants $\{c_0, \ldots c_n\}$ s.t. $c_0\varphi_0(x) + \ldots + c_n\varphi_n(x) = 0 \ \forall x$, then $c_0 = c_1 = \ldots = c_n = 0$.

(style points) 3. (Help GSI remember who you are) What is your favorite plant or animal?

animal: cat

plant: bristlecone pine

Prepared by ANNA LIEB. Comments and questions: **lieb@math.berkeley.edu**