MT-630, DIFFERENTIAL GEOMETRY.

Problem 0.1. We say that a subset $l \subset \mathbb{R}^2$ is a *non-vertical affine line* if there exists constants $a_l, b_l \in \mathbb{R}$ such that

$$l = \{(x, y) \in \mathbb{R}^2 \mid y = a_l x + b_l\}. \tag{0.1a}$$

Similarly, we say that a subset $l \subset \mathbb{R}^2$ is a *non-horizontal affine line* if there exists constants $c_l, d_l \in \mathbb{R}$ such that

$$l = \{(x, y) \in \mathbb{R}^2 \mid x = c_l y + d_l\}. \tag{0.1b}$$

A subset $l \subset \mathbb{R}^2$ is simply called an *affine line* if it is either of the form (0.1a) or (0.1b).

Suppose that $l \subset \mathbb{R}^2$ is an affine line, we say that a point $(x_0, y_0) \in \mathbb{R}^2$ is on l if $(x_0, y_0) \in l$. For example, if $l = \{(x, y) \in \mathbb{R}^2 \mid y = a_l x + b_l\}$ is non-vertical, (x_0, y_0) is on l if and only if $y_0 = a_l x_0 + b_l$.

Let

$$M = \{\big(l, (x_l, y_l)\big) \mid l \subset \mathbb{R}^2 \text{ is an affine line, and } (x_l, y_l) \in l \text{ is a point on that line}\}.$$

Consider the charts $\{(U, \phi), (V, \psi)\}$ for M given by

$$\begin{aligned}
U &:= \Big\{\big(l, (x_l, y_l)\big) \in M \Big| l \text{ is a non-horizontal affine line}\Big\} \\
\phi : U \to \mathbb{R}^3, &\qquad \phi\big(l, (x_l, y_l)\big) = (c_l, d_l, y_l), \\
&\text{where } l = \{(x, y) \in \mathbb{R}^2 \mid x = c_l y + d_l\}
\end{aligned}$$

$$\begin{aligned}
V &:= \Big\{\big(l, (x_l, y_l)\big) \in M \Big| l \text{ is a non-vertical affine line}\Big\} \\
\psi : V \to \mathbb{R}^3, &\qquad \psi\big(l, (x_l, y_l)\big) = (a_l, b_l, x_l), \\
&\text{where } l = \{(x, y) \in \mathbb{R}^2 \mid y = a_l x + b_l\}
\end{aligned}$$

(1) Give a formula for the transition map

$$\phi \circ \psi^{-1} : \psi(V) \to \phi(U).$$

(2) Show that the transition map $\phi \circ \psi^{-1}$ is smooth.

Since $\phi : U \to \mathbb{R}^3$ and $\psi : V \to \mathbb{R}^3$ are bijective, this implies that M is a 3-dimensional manifold.

Problem 0.2. As in the previous problem, let M be the set of pairs $(l, (x_l, y_l))$ where $l \subset \mathbb{R}^2$ is an affine line and $(x_l, y_l) \in l$ is a point on the line l. Consider the map

$$P : \mathbb{R}^3 \to M$$

which sends a triple $(x, y, \theta) \in \mathbb{R}^3$ to $(l_\theta, (x, y))$, where l_θ is the line through (x, y) which makes an angle of θ with the positive x-axis.

(1) Give a formula for P with respect to each of the charts $\{(U, \phi), (V, \psi)\}$ for M,
(2) Show that $P : \mathbb{R}^3 \to M$ is smooth.
(3) Is P a bijection? If so, provide a proof; if not, provide a counter example.