

Exercises (Week 2)

1) Recall that a topological space X is compact if for any collection of open sets $\mathcal{C} = \{U_\alpha \subseteq X\}$ which covers X (i.e. $X \in \bigcup_{U_\alpha \in \mathcal{C}} U_\alpha$), there is a finite subcollection $U_{\alpha_1}, \dots, U_{\alpha_n}$ which already covers X :

$$X \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$$

Prove the Heine-Borel Theorem

Thm (Heine-Borel) For any $a < b \in \mathbb{R}$, the closed interval $[a, b] \subset \mathbb{R}$, equipped with the standard topology, is compact.

Hint: Let $A = \left\{ x \in [a, b] \mid \begin{array}{l} [a, x] \text{ is covered by} \\ \text{a finite number of elements} \end{array} \right\} \subseteq [a, b]$

& prove the following lemmas

Lemma 1 A is non-empty (show that $a \in A$)

Lemma 2 if $x \in A$ & $x < b$, then for $\varepsilon > 0$ sufficiently small, $x + \varepsilon \in A$ (Hint, show that $[x, x + \varepsilon]$ is contained in a single element of \mathcal{C})

Lemma 3 $\sup(A) \in A$

Lemma 4 $\sup(A) = b$

Exercise 2 Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is smooth & bijective with $f'(<) \neq 0$. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ denote the inverse of f . Show:

$$1) \quad g'(y) = \frac{1}{f'(x)} \quad \text{with } x = g(y)$$

$$2) \quad g''(y) = \frac{-f''(x)}{f'(x)^3} \quad \text{with } x = g(y)$$

$$3) \quad g'''(y) = \frac{-f'''(x)}{f'(x)^4} + 3 \frac{f''(x)^2}{f'(x)^5} \quad \text{with } x = g(y)$$

Notice that only $f'(x)$ appears in the denominator & argue that g is 3-times differentiable. What do you expect for higher derivatives of g ? Do you expect g to be smooth?

Exercise 3 Let $U \subseteq \mathbb{R}^l$, $V \subseteq \mathbb{R}^m$, $W \subseteq \mathbb{R}^n$ be open sets & $f: U \rightarrow V$, $g: V \rightarrow W$ smooth functions.

Define linear maps (matrices)

$$Df(x): \mathbb{R}^l \rightarrow \mathbb{R}^m, Dg(y): \mathbb{R}^m \rightarrow \mathbb{R}^n$$

$$\text{&} D(g \circ f)(x): \mathbb{R}^l \rightarrow \mathbb{R}^n$$

by $Df(x) = \begin{pmatrix} \frac{\partial f^1}{\partial x^1}(x) & \dots & \frac{\partial f^1}{\partial x^l}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1}(x) & \dots & \frac{\partial f^m}{\partial x^l}(x) \end{pmatrix}$

$$Dg(y) = \begin{pmatrix} \frac{\partial g^1}{\partial y^1}(y) & \dots & \frac{\partial g^1}{\partial y^m}(y) \\ \frac{\partial g^2}{\partial y^1}(y) & \dots & \frac{\partial g^2}{\partial y^m}(y) \\ \vdots & \ddots & \vdots \\ \frac{\partial g^n}{\partial y^1}(y) & \dots & \frac{\partial g^n}{\partial y^m}(y) \end{pmatrix}$$

$$D(g \circ f)(x) = \begin{pmatrix} \frac{\partial(g \circ f)^1}{\partial x^1}(x) & \dots & \frac{\partial(g \circ f)^1}{\partial x^l}(x) \\ \frac{\partial(g \circ f)^2}{\partial x^1}(x) & \dots & \frac{\partial(g \circ f)^2}{\partial x^l}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial(g \circ f)^n}{\partial x^1}(x) & \dots & \frac{\partial(g \circ f)^n}{\partial x^l}(x) \end{pmatrix}$$

& prove $D(g \circ f)(x) = Dg(f(x)) Df(x)$

where the right hand side denotes matrix multiplication.

Exercise 4

Let $f(x, y, z) = \cos(x)\sin(y) + \cos(y)\sin(z) + \cos(z)\sin(x)$

i) Prove that

$$0 \neq Df(x, y, z) := \left(\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z) \right)$$

whenever $f(x, y, z) = 0$

Hint: Show $Df(x, y, z) = 0$ is equivalent to the following three equations:

$$A) \quad \sin(x)\sin(y) = \cos(z)\cos(x)$$

$$B) \quad \sin(y)\sin(z) = \cos(x)\cos(y)$$

$$C) \quad \sin(z)\sin(x) = \cos(y)\cos(z)$$

& Consider the equations

$$\frac{AB}{C} \Leftrightarrow \left(\sin^2(y) = \cos^2(y) \right), \quad \frac{BC}{A} \Leftrightarrow \left(? = ? \right) \text{ & } \frac{CA}{B} \Leftrightarrow \left(? = ? \right)$$

ii) Part i) implies that $\{(x, y, z) \mid f(x, y, z) = 0\}$ is a 2-dimensional manifold (which approximates the gyroid). Plot it using your favorite software package (SageMathCloud, for example).