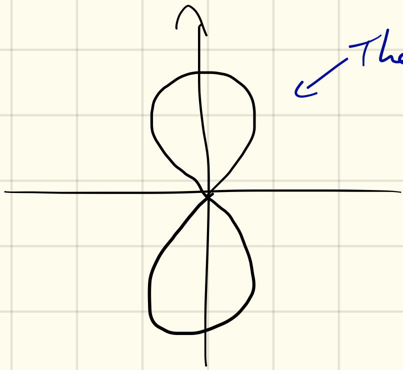
Exercises:

1) Let $(0, 2\pi)_{std}$ denote the interval $\{0 < x < 2\pi\}$ equipped with the standard topology,

& let $(0, 2\pi)_8$ denote the same set equipped with the subspace topology arising from the map

$$i : (0, 2\pi)_8 \longrightarrow \mathbb{R}^2$$
$$i : t \longrightarrow (\sin^2(t)\cos(t), \sin^3(t))$$



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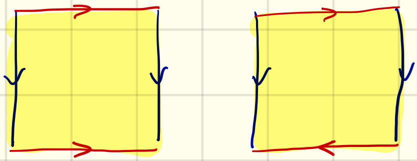
Show that the idenity map $(0, 2\pi)_8 \longrightarrow (0, 2\pi)_{std}$, $t \longrightarrow t$, is not continuous.

2) Prove that the function

$$f(x)=\begin{cases} e^{-\frac{1}{x}} & \text{for } x>0 \\ 0 & \text{for } x\leq 0 \end{cases}$$

is continuous.

3) A convenient way of constructing surfaces is through gluing diagrams. For instance, the following gluing diagrams represent a torus and a klein bottle, respectively: Edges with the same label are to be glued, with the orientation indicated by the arrow:



Using gluing diagrams, explain that it is possible to cut the klein bottle along a circle in such a way that the resulting surface is

a) a cylinder (with two boundary circles),

b) a Möbius strip, or

c) two Möbius strips.

4) The space of affine lines in $\mathbb{R}^2$ has an atlas given by the pair of coordinate charts

$U =$ 'Non vertical lines' $\xrightarrow{\varphi} \mathbb{R}^2$

$\varphi : \{y = ax + b\} \longrightarrow (a,b) \in \mathbb{R}^2$

$V =$ 'non-horizontal lines' $\xrightarrow{\psi} \mathbb{R}^2$

$\psi : \{x = cy + d\} \longrightarrow (c,d) \in \mathbb{R}^2$

with transition maps $\psi \circ \varphi^{-1} : (a,b) \longrightarrow \left(c = \frac{1}{a}, d = \frac{-b}{a}\right)$

& $\varphi \circ \psi^{-1} : (c,d) \longrightarrow \left(a = \frac{1}{c}, b = \frac{-d}{c}\right)$

Determine which 'standard surface' the resulting manifold is (e.g. 'Sphere', 'Torus', 'Mobius strip', 'cylinder', etc.) & explain why.

5) Let $N=(0,\dots,0,1)\in\mathbb{R}^{n+1}$ be the 'north pole' & $(0,\dots,0,-1)$ be the 'south pole' of the sphere

$$S^n=\{(x^0,\dots,x^n)\in\mathbb{R}^{n+1} \mid (x^0)^2+\cdots+(x^n)^2=1\}$$

Define stereographic projection $\sigma: S^n\setminus\{N\}\to\mathbb{R}^n$ by

$$\sigma(x^0,\dots,x^n)=\frac{(x^0,\dots,x^{n-1})}{1-x^n}$$

& let $\tilde{\sigma}=\sigma(-x)$ for $x\in S^n\setminus\{S\}$.

a) Show that σ is bijective and

$$\sigma^{-1}(u^1,\dots,u^n)=\frac{(2u^1,\dots,2u^n,(u^1)^2+\cdots+(u^n)^2-1)}{(u^1)^2+\cdots+(u^n)^2+1}$$

b) Compute the transition map $\tilde{\sigma}\circ\sigma^{-1}$ and verify that the two charts $(S^n\setminus\{N\},\sigma)$ and $(S^n\setminus\{S\},\tilde{\sigma})$ are compatible and define an atlas on S^n.

c) Verify that each of these charts are compatible with the ones we gave in class: $(U_i^{\pm},\varphi_i^{\pm})$

$U_i^{\pm}=\{(x^0,\dots,x^n) \mid \pm x^i>0\}$ $\quad\varphi_i^{\pm}(x^0,\dots,x^n)=(x^0,\dots,x^{i-1},x^{i+1},\dots,x^n)$