MATH 189, MATHEMATICAL METHODS IN CLASSICAL AND QUANTUM MECHANICS.
HOMEWORK 5. DUE WEDNESDAY, OCTOBER 15TH, 2014

HTTP://MATH.BERKELEY.EDU/~LIBLAND/MATH-189/HOMEWORK.HTML

In this problem sets, we will study the equations of motion for a rigid body in the absence of external forces, from the perspective of geodesics on O(3).

Exercise 1. Let I be the moment of inertia for the rigid body (a 3×3-matrix). Recall that the kinetic energy of the rigid body is

$$K = \frac{1}{2}\vec{\omega} \cdot I(\vec{\omega}),$$

where ω is the angular velocity. Recall that the moment of inertia, I is constant, when expressed in body coordinates.

Since there are no forces, and hence no potential energy, at first glance, one would think the Lagrangian is $L(\vec{\omega}) = K = \frac{1}{2}\vec{\omega} \cdot I(\vec{\omega})$.

Write down the Euler Lagrange equations for this Lagrangian, and compute the corresponding equations of motion.

Are these the correct equations of motion? If not, find the error. *Hint: The Euler Lagrange equations hold if and only if the action is stationary for variations of $\vec{\omega}(t)$ which fix the values of $\vec{\omega}(t_0)$ and $\vec{\omega}(t_1)$. Are these the correct variations to consider? Remember what $\vec{\omega}$ is defined to be...*

Exercise 2. Let $\Omega = [\vec{\omega}]_\times$. Show that for any moment of inertia I, there exists a unique symmetric 3×3-matrix A, such that

$$\mathrm{tr}(\Omega A \Omega) = \frac{1}{2}\vec{\omega} \cdot I(\vec{\omega})$$

whenever $\Omega = [\vec{\omega}]_\times$. Note $\mathrm{tr}(M)$ is the trace of the matrix M.

1. Geodesics on O(3)

The map $\Omega \to A\Omega$ is a linear map. Moreover, $\mathrm{tr}(B^*C)$ is the standard innerproduct of two matrices B and C. Since the map $\Omega \to A\Omega$ is symmetric, we may interpret $\Omega \to \|\Omega\|_A^2 := \mathrm{tr}(\Omega A \Omega)$ as defining a Riemannian metric on the space of matrices (in this case, it is actually just an innerproduct on the vector space of 3×3-matrices).

Thus, we may rewrite the Lagrangian as

$$L(\Omega) = \mathrm{tr}(\Omega A \Omega),$$

where $\Omega = [\vec{\omega}]_\times$, and $\vec{\omega}$ is the angular velocity as measured in body coordinates.

Now let R_t be the 3×3 matrix which maps the rigid body at time 0 to it's orientation at time t (i.e. $R_t \in \mathrm{O}(3)$ is a rotation matrix). Then $\vec{\omega} = \overrightarrow{[R_t^{-1}\dot{R}_t]}$, or $\Omega = R_t^{-1}\dot{R}_t = R_t^*\dot{R}_t$. Therefore $L(R, \dot{R}) = \mathrm{tr}(R^*\dot{R}AR^*\dot{R})$.

We define a Riemannian metric on the vector space $\mathbb{R}^{3\times 3}$ of 3×3-matrices at $X \in \mathbb{R}^{3\times 3}$ to be

$$\|\dot{X}\|^2(X) = \mathrm{tr}(X^*\dot{X}AX^*\dot{X})$$

for the matrix $\dot{X} \in \mathbb{R}^{3\times 3}$. Then the solutions to the Euler-Lagrange equations for

$$L(R, \dot{R}) = \mathrm{tr}(R^*\dot{R}AR^*\dot{R}).$$

are just the geodesics in $\mathbb{R}^{3\times 3}$.

Exercise 3. Recall that $R \in \mathrm{O}(3)$ is subject to the constraint $R^*R = 1$. Thus the constrained Lagrangian is

$$L'(R, \dot{R}, \lambda) := \mathrm{tr}(R^*\dot{R}AR^*\dot{R}) - \mathrm{tr}(\lambda(R^*R - 1)),$$

where R, $\dot{R}$ and λ are now allowed to be arbitrary 3×3-matrices.

Show that the corresponding Euler Lagrange equations are

$$\dot{\Pi} = 2\Pi\Omega - \lambda^* - \lambda.$$

where $\Omega = R^*\dot{R}$, and $\Pi = A\Omega + \Omega A$.

(Here Π is the generalized momentum corresponding to Ω.)

Hint: you may need to use the identities $\mathrm{tr}(BC) = \mathrm{tr}(CB)$ and $\mathrm{tr}(B^*) = \mathrm{tr}(B)$ to reduce the Euler-Lagrange equations to the above.

Next, notice that $\dot{\Pi}$ is skew symmetric, while $-\lambda^* - \lambda$ is symmetric to reduce the above equation to

$$\dot{\Pi} = -[\Omega, \Pi].$$

Here $[\Omega, \Pi] = \Omega\Pi - \Pi\Omega$ is the commutator.

The equation

$$\dot{\Pi} = -[\Omega, \Pi].$$

is both the equation for geodesics on $\mathrm{O}(3)$, and the equation of motion for the rigid body.