MATH 189, MATHEMATICAL METHODS IN CLASSICAL AND QUANTUM MECHANICS.
HOMEWORK 2. DUE WEDNESDAY SEPTEMBER 10TH, 2014

HTTP://MATH.BERKELEY.EDU/~LIBLAND/MATH-189/HOMEWORK.HTML

0.1. **Moment of Inertia.** Recall that the i^{th} particle, at position $\vec{r_i}$ with mass m_i contributes the operator

$$I_i : \vec{\omega} \to m_i \vec{r_i} \times (\vec{\omega} \times \vec{r_i})$$

to the total moment of inertia operator $I := \sum_i I_i$.

Define $\mathbf{S}_{\vec{r}}$ to be the symmetric operator

$$\mathbf{S}_{\vec{r}} : \vec{\omega} \to \vec{r} \times (\vec{\omega} \times \vec{r}),$$

so that $I_i = m_i \mathbf{S}_{\vec{r_i}}$.

Exercise 1. Compute the matrix representation for the symmetric operator $\mathbf{S}_{\vec{r}}$, in terms of $\vec{r} = (x, y, z)$. Compute the matrix representation for the symmetric operator I_i, in terms of m_i and $\vec{r_i} = (x_i, y_i, z_i)$.

Exercise 2. Compute the moment of inertia for a text book. Assume that the textbook is w units wide, h units tall, and p units thick, and that it occupies the region

$$R = \{(x, y, z) \mid |x| \leq w/2, |y| \leq h/2, |z| \leq p/2\}.$$

Moreover, assume that the textbook is of uniform density and mass M. Thus a infinitesimal volume of the text book, $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ has mass

$$\mathrm{d}m := \frac{M}{whp}\mathrm{d}V,$$

and contributes

$$\mathrm{d}I := \mathbf{S}_{(x,y,z)}\mathrm{d}m$$

to the total moment of inertia for the book:

$$I^{\mathrm{book}} := \int_R \mathrm{d}I.$$

Compute the matrix representation for I^{book} in the standard coordinates.

1. Infinitesimal Rotations

The set of distance preserving linear transformations of $\mathbb{R}^3$,

$$\mathrm{O}(3) = \{R \text{ a } 3 \times 3 \text{ matrix} \mid R^* R = \mathbf{1}\},$$

is called the *orthogonal group* (here R^* is the transpose of R and $\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the identity).

Exercise 3. Suppose $R_t = \mathbf{1} + X_1 t + X_2 t^2 + \cdots$ is the taylor expansion for a path of matrices $R_t \in \mathrm{O}(3)$. Using the fact that $R_t^* R_t = \mathbf{1}$ for all t, show that X_1 is a skew-symmetric matrix (i.e. $X_1^* + X_1 = \mathbf{0}$). (Hint: In the equation $R_t^* R_t = \mathbf{1}$, you only need to compare the terms which are first-order in t).

Conclude that, to first order, a path of rotations is given by a skew-symmetric matrix.

Thus, the infinitesimal rotations of Euclidean 3-space can be identified with the space of skew-symmetric matrices:

$$\mathfrak{o}(3) = \{X \text{ a } 3 \times 3 \text{ matrix} \mid X^* + X = \mathbf{0}\},$$

For any vector $\vec{\omega} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, we defined the linear transformation $[\vec{\omega}]_\times : \mathbb{R}^3 \to \mathbb{R}^3$ by the cross product with $\vec{\omega}$:

$$[\vec{\omega}]_\times(\vec{r}) := \vec{\omega} \times \vec{r}.$$

Conversely, for any 3×3-skew-symmetric matrix $X \in \mathfrak{o}(3)$, we defined the vector $\overrightarrow{[X]} \in \mathbb{R}^3$ to be the unique vector such that $X \equiv \left[\overrightarrow{[X]}\right]_\times$.

Suppose $R_t = \mathbf{1} + X_1 t + X_2 t^2 + \cdots$ is the taylor expansion for a path of rotations $R_t \in \mathrm{O}(3)$, with $X_1 = [\vec{\omega}]_\times$.

Exercise 4. Show that

$$\left.\frac{\partial R_t}{\partial t}\right|_{t=0} \vec{r} = \vec{\omega} \times \vec{r}.$$

Conclude that the rotation R_t is given (to first order in t) by the cross-product with $\vec{\omega}$.

The instantaneous axis of rotation for R_t at time $t = 0$ is defined to be the set points which are fixed by $\left.\frac{\partial R_t}{\partial t}\right|_{t=0}$, i.e. the vector space

$$\{\vec{r} \mid \left.\frac{\partial R_t}{\partial t}\right|_{t=0} \vec{r} = 0\}$$

Exercise 5. Show that, when $\vec{\omega} \neq 0$, the instantaeous axis of rotation at $t = 0$ is spanned by $\vec{\omega}$.

Let $\vec{e}$ be a unit vector which is perpendicular to the axis of rotation, and define $\vec{e}_t := R_t \vec{e}$. The instantaneous angular speed of the time-dependant rotation R_t at time $t = 0$ is defined to be the speed $\|\dot{\vec{e}}_0\|$ at which $\vec{e}_t$ moves at time $t = 0$.

Exercise 6. Show that the instantaneous angular speed of the time-dependant rotation R_t at time $t = 0$ is $\|\vec{\omega}\|$.

In conclusion, the vector $\vec{\omega}$ can be seen as describing (via the cross product) an infinitesimal rotation with angular speed $\|\vec{\omega}\|$ about the axis spanned by $\vec{\omega}$. The direction of the rotation is given by the 'right hand rule' (`http://en.wikipedia.org/wiki/Right-hand_rule`). As a result we call ω the angular velocity of the time-dependant rotation R_t at time $t = 0$.

Suppose we have a rigid body whose position at time $t = 0$ is $U_0 \subseteq \mathbb{R}^3$, and which is rotating about the origin[1]. A priori, it is not obvious that the rigid body has an instantaneous angular velocity or even an instantaneous axis of rotation (why should it)?

However, it is not challenging to show that there exists a time-dependent rotation

$$R_t = \mathbf{1} + X_1 t + X_2 t^2 + \cdots \in \mathrm{O}(3)$$

such that the position of the rigid body at any other time $t \neq 0$ is $R_t(U_0)$. A consequence of these exercises is that $\vec{\omega} := \overrightarrow{[X_1]}$ is the instantaneous angular velocity of the rigid body. In particular, *the instantaneous angular velocity* is a well defined concept for a rigid body rotating about its centre of mass.

Exercise 7. For any orthogonal matrix $R \in \mathrm{O}(3)$, and any vector $\vec{\omega} \in \mathbb{R}^3$, show that $R[\vec{\omega}]_\times R^{-1} = [R\vec{\omega}]_\times$.

2. Equations of motion for a Rigid body

Consider a rigid body $B \subset \mathbb{R}^3$ with center of mass fixed at the origin of $\mathbb{R}^3$. Let $R_t : \mathbb{R}^3 \to \mathbb{R}^3$ be the matrix which maps the body at time 0 to its position at time t. We have the following chart of quantities as measured in the two coordinate systems:

	Internal Coordinates (relative to $\{R_t\hat{\imath}, R_t\hat{\jmath}, R_t\hat{k}\}$)	External Coordinates (relative to $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$)
The rigid body	B	$B_t = R_t(B)$
Position of a particle of the body	$\vec{r}$	$\vec{r}_t = R_t\vec{r}$
Moment of inertia	I^b	$R_t I^b R_t^{-1}$
Angular velocity	$\vec{\omega}_t^b = R_t^{-1}\vec{\omega}_t$	$\vec{\omega}_t = \overrightarrow{[\dot{R}_t R_t^{-1}]}$
Angular momentum	$\vec{L}_t^b$	$\vec{L}_t = R_t \vec{L}_t^b$
External torque	$\vec{\tau}_t^b$	$\vec{\tau}_t = R_t \vec{\tau}_t^b$

Exercise 8. Prove that $R_t I^b \vec{\omega}_t^b = \vec{L}_t$, and conclude that $R_t I^b \vec{\omega}_t^b$ is constant in the absence of external torques.

Exercise 9. Using Exercise 7, prove that $[\vec{\omega}_t^b]_\times = R_t^{-1}\dot{R}_t$.

Exercise 10 (Equations of motion for a rigid body, in the absence of external torques.)**.** Prove that $\vec{\omega}_t^b$ satisfies the 1^{st}-order ODE:

$$I^b \dot{\vec{\omega}}_t^b = -\vec{\omega}_t^b \times (I^b \vec{\omega}_t^b). \tag{2.1}$$

Hint: Use the Leibniz rule to compute the time-derivative of $R_t I^b \vec{\omega}_t^b = \vec{L}$, and simplify the resulting expression using Exercise 9.

Note that to obtain the equation of motion (2.1) for a rigid body, you did not use Newton's laws explicitly. Instead you only needed to know the conserved quantities in the problem.

[1] for example, if the origin is it's center of mass, and the sum of the external forces is zero