MATH 189, MATHEMATICAL METHODS IN CLASSICAL AND QUANTUM MECHANICS.
HOMEWORK 2. DUE WEDNESDAY OCTOBER 8TH, 2014

HTTP://MATH.BERKELEY.EDU/~LIBLAND/MATH-189/HOMEWORK.HTML

1. Lagrange Multipliers (Easy)

Exercise 1 (Not to be graded)**.** In each of the following questions, compute the local extremas for f, subject to the constraints $g_i = c_i$.

(1) $f(x,y) = x^2y$, $g(x,y) = x^2 + y^2$, $c = 3$.
(2) $f(p_1,\ldots,p_n) = -\sum_{i=1}^n p_i \log_2(p_i)$, $g(p_1,\ldots,p_n) = \sum_{i=1}^n p_i$, $c = 1$.
(3) $f(x,y) = 3x^2 + 5y^3$, $g(x,y) = x^2 - y^3$, $c = 1$
(4) $f(x,y) = x^2 - 3y^2$, $g_1(x,y,z) = x + 2y + 3z$, $g_2(x,y,z) = x^2 + y^2$, $c_1 = 0$, $c_2 = 1$

Exercise 2. Find the local minimas and maximas of $f(x,y) = 4x^2 + 10y^2$ subject to the inequality $x^2 + y^2 \leq 4$.

Note that f will attain both it's maximum and minimum in the region $x^2 + y^2 \leq 4$, since this region is compact. So you will find at least two points: one will correspond to the minimum, and the other to the maximum.

2. Euclidean Geometry (for personal edification, not to be graded)

Exercise 3 (Not to be graded)**.** Consider a free particle of mass $m = 1$ in 3-dimensional Euclidean space. It will have the Lagrangian

$$L(\vec{q},\dot{\vec{q}}) = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2},$$

where $\vec{q} = (x,y,z)$ and $\dot{\vec{q}} = (\dot{x},\dot{y},\dot{z})$.

Now suppose that the particle is subject to the constraint $g(\vec{q}) = 1$, for $g(\vec{q}) = z$.

Part A: What is the constrained Lagrangian, $L_{constr}(x,y,z,\dot{x},\dot{y},\dot{z},\lambda)$? Write down the Euler-Lagrange equations for the constrained Lagrangian.

Part B: Show that

(1) $\Phi_s^{\hat{i}}(x,y,z) = (x, y-s, z)$,
(2) $\Phi_s^{\hat{j}}(x,y,z) = (x+s, y, z)$, and
(3) $\Phi_s^{\hat{k}}(x,y,z) = (\cos(s)x - \sin(s)y, \sin(s)x + \cos(s)y, z)$,

are all symmetries of the constrained Lagrangian, that is:

$$(\mathrm{d}\Phi_s^{\hat{i}})^* L_{constr} = L_{constr}, \quad (\mathrm{d}\Phi_s^{\hat{j}})^* L_{constr} = L_{constr}, \quad (\mathrm{d}\Phi_s^{\hat{k}})^* L_{constr} = L_{constr}.$$

Denote the conserved quantities corresponding to $\Phi_s^{\hat{\imath}}$, $\Phi_s^{\hat{\jmath}}$ and $\Phi_s^{\hat{k}}$ by $Q^{\hat{\imath}}$, $Q^{\hat{\jmath}}$, and $Q^{\hat{k}}$. We define the vector $\vec{Q}$ as

$$\vec{Q} = (Q^{\hat{\imath}}, Q^{\hat{\jmath}}, Q^{\hat{k}}).$$

Show that the equality

$$\vec{Q} = \vec{q} \times \dot{\vec{q}}$$

holds when both sides are evaluated at points satisfying constraints: $g(\vec{q}) = 1$ and $\nabla g \cdot \dot{\vec{q}} = 0$.

Conclude that $\vec{q} \times \dot{\vec{q}}$ is a conserved quantity.

Part C: Using the fact that $\vec{q} \times \dot{\vec{q}} = \vec{L}$ is conserved, show that $\vec{q}$ lies in the plane

$$\Pi_{\vec{L}} := \{\vec{q} \mid \vec{q} \cdot \vec{L} = 0\}$$

for all time, as it evolves according to the constrained equations of motion.

Prove that the path the particle traces as it evolves according to the constrained equations of motion is

$$\vec{q} \in \Pi_{\vec{L}} \cap g^{-1}(1),$$

where

$$g^{-1}(1) = \{\vec{q} \mid g(\vec{q}) = 1\}.$$

Graph[1] both the level set $g(\vec{x}) = 1$ and the plane $\vec{x} \cdot \vec{L} = 0$ in the same picture, where $\vec{L} = (2, 0, 1)$. Visualize the intersection $\Pi_{\vec{L}} \cap g^{-1}(1)$.

Part D: What is the Hamiltonian corresponding to the constrained Lagrangian? Using the conservation of energy, compute the explicit trajectory $\vec{f}_{0,\phi}(t)$ of the particle when

$$\frac{\vec{L}}{\|\vec{L}\|} = \big(\cos(\phi), 0, -\sin(\phi)\big)$$

and the energy is $\frac{1}{2}$.

Using the fact that $\Phi_s^{\hat{k}}$ is a symmetry, show that

$$\vec{f}_{\psi,\phi} : t \to \Phi_\psi^{\hat{k}} \vec{f}_{0,\phi}(t)$$

is a trajectory of the particle when

$$\frac{\vec{L}}{\|\vec{L}\|} = \big(\cos(\psi)\cos(\phi), \sin(\psi)\cos(\phi), -\sin(\phi)\big)$$

and the energy is $\frac{1}{2}$.

[1] You may use Mathematica, or Sage `http://math.berkeley.edu/~libland/math-189/useful-resources/`

Part E: We may project the level set $\mathfrak{g}^{-1}(1)$ to the x-y-plane using the stereographic projection through the point $(0,0,-1)$:

$$\mathrm{St}:(x,y,z)\to\frac{1}{1+z}(x,y).$$

Plot the images of the trajectories $\mathrm{St}\circ\vec{f}_{\psi,\phi}$ for the values $\psi=0,\frac{2\pi}{3},\frac{4\pi}{3}$, and $\phi=0,\frac{\pi}{8},\frac{2\pi}{8},\frac{3\pi}{8},\dots,\frac{4\pi}{8}$.

3. Elliptic Geometry (For personal edification, not to be graded)

Exercise 4. Consider a free particle of mass $m=1$ in 3-dimensional Euclidean space. It will have the Lagrangian

$$L(\vec{q},\dot{\vec{q}})=\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2},$$

where $\vec{q}=(x,y,z)$ and $\dot{\vec{q}}=(\dot{x},\dot{y},\dot{z})$.

Now suppose that the particle is subject to the constraint $g(\vec{q})=1$, for $g(\vec{q})=x^2+y^2+z^2$.

Part A: What is the constrained Lagrangian, $L_{constr}(x,y,z,\dot{x},\dot{y},\dot{z},\lambda)$? Write down the Euler-Lagrange equations for the constrained Lagrangian.

Part B: Show that

(1) $\Phi^{\hat{\imath}}_s\vec{q}=\begin{pmatrix}1&0&0\\0&\cos(s)&-\sin(s)\\0&\sin(s)&\cos(s)\end{pmatrix}\vec{q}$,

(2) $\Phi^{\hat{\jmath}}_s\vec{q}=\begin{pmatrix}\cos(s)&0&\sin(s)\\0&1&0\\-\sin(s)&0&\cos(s)\end{pmatrix}\vec{q}$, and

(3) $\Phi^{\hat{k}}_s\vec{q}=\begin{pmatrix}\cos(s)&-\sin(s)&0\\\sin(s)&\cos(s)&0\\0&0&1\end{pmatrix}\vec{q}$,

are all symmetries of the constrained Lagrangian. Denote the conserved quantities corresponding to $\Phi^{\hat{\imath}}_s$, $\Phi^{\hat{\jmath}}_s$ and $\Phi^{\hat{k}}_s$ by $Q^{\hat{\imath}}$, $Q^{\hat{\jmath}}$, and $Q^{\hat{k}}$. We define the vector $\vec{Q}$ as

$$\vec{Q}=(Q^{\hat{\imath}},Q^{\hat{\jmath}},Q^{\hat{k}}).$$

Show that the equality

$$\vec{Q}=\vec{q}\times\dot{\vec{q}}$$

holds.

Conclude that $\vec{q}\times\dot{\vec{q}}$ is a conserved quantity.

Part C: Using the fact that $\vec{q}\times\dot{\vec{q}}=\vec{L}$ is conserved, show that $\vec{q}$ lies in the plane

$$\Pi_{\vec{L}}:=\{\vec{q}\mid\vec{q}\cdot\vec{L}=0\}$$

for all time, as it evolves according to the constrained equations of motion.

Prove that the path the particle traces as it evolves according to the constrained equations of motion is

$$\vec{q} \in \Pi_{\vec{L}} \cap g^{-1}(1).$$

Graph both the level set $g(\vec{x}) = 1$ and the plane $\vec{x} \cdot \vec{L} = 0$ in the same picture, where $\vec{L} = (2, 0, 1)$. Visualize the intersection $\Pi_{\vec{L}} \cap g^{-1}(1)$.

Part D: What is the Hamiltonian corresponding to the constrained Lagrangian? Using the conservation of energy, compute the explicit trajectory $\vec{f}_{\psi,\phi}(t)$ of the particle when

$$\frac{\vec{L}}{\|\vec{L}\|} = \big(\cos(\psi)\cos(\phi), \sin(\psi)\cos(\phi), -\sin(\phi)\big)$$

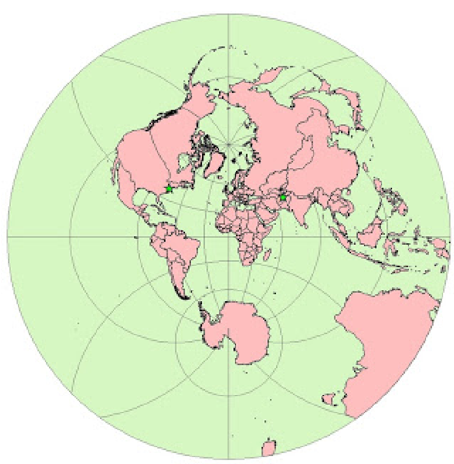
and the energy is $\frac{1}{2}$.

Hint: First compute the trajectory $\vec{f}_{0,\phi}$ when $\psi = 0$, and then use the symmetry $\Phi_s^{\hat{k}}$ to map it to the trajectory $\vec{f}_{\psi,\phi}$ for an arbitrary ψ.

Part E: We may project the level set $\mathfrak{g}^{-1}(1)$ to the x-y-plane using the stereographic projection through the point $(0, 0, -1)$:

$$\mathrm{St} : (x, y, z) \to \frac{1}{1+z}(x, y).$$

Plot the images of the trajectories $\mathrm{St} \circ \vec{f}_{\psi,\phi}$ for the values $\psi = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$, and $\phi = 0, \frac{\pi}{8}, \frac{2\pi}{8}, \frac{3\pi}{8}, \ldots, \frac{4\pi}{8}$.



4. Hyperbolic Geometry (To be graded)

Exercise 5. Consider a free particle of mass $m = 1$ in (2+1)-dimensional Lorentzian space. It will have the Lagrangian

$$L(\vec{q}, \dot{\vec{q}}) = \frac{\dot{x}^2 + \dot{y}^2 - \dot{z}^2}{2},$$

where $\vec{q} = (x, y, z)$ and $\dot{\vec{q}} = (\dot{x}, \dot{y}, \dot{z})$.

Now suppose that the particle is subject to the constraints $z > 0$ and $g(\vec{q}) = 1$, for $g(\vec{q}) = -x^2 - y^2 + z^2$.

Part A: What is the constrained Lagrangian, $L_{constr}(x, y, z, \dot{x}, \dot{y}, \dot{z}, \lambda)$? Write down the Euler-Lagrange equations for the constrained Lagrangian.

Part B: Show that

(1) $\Phi_s^{\hat{\imath}}\vec{q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(s) & -\sinh(s) \\ 0 & -\sinh(s) & \cosh(s) \end{pmatrix} \vec{q}$,

(2) $\Phi_s^{\hat{\jmath}}\vec{q} = \begin{pmatrix} \cosh(s) & 0 & \sinh(s) \\ 0 & 1 & 0 \\ \sinh(s) & 0 & \cosh(s) \end{pmatrix} \vec{q}$, and

(3) $\Phi_s^{\hat{k}}\vec{q} = \begin{pmatrix} \cos(s) & -\sin(s) & 0 \\ \sin(s) & \cos(s) & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{q}$,

are all symmetries of the constrained Lagrangian. Denote the conserved quantities corresponding to $\Phi_s^{\hat{\imath}}$, $\Phi_s^{\hat{\jmath}}$ and $\Phi_s^{\hat{k}}$ by $Q^{\hat{\imath}}$, $Q^{\hat{\jmath}}$, and $Q^{\hat{k}}$. We define the vector $\vec{Q}$ as

$$\vec{Q} = (Q^{\hat{\imath}}, Q^{\hat{\jmath}}, Q^{\hat{k}}).$$

Show that the equality

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holds.

Conclude that $\vec{q} \times \dot{\vec{q}}$ is a conserved quantity.

Part C: Using the fact that $\vec{q} \times \dot{\vec{q}} = \vec{L}$ is conserved, show that $\vec{q}$ lies in the plane

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[2] You may use Mathematica, or Sage `http://math.berkeley.edu/~libland/math-189/useful-resources/`, or do it by hand. There are also some online tools, such as `http://www.archimy.com` which can be used if you don't want to install any software.

Part D: What is the Hamiltonian corresponding to the constrained Lagrangian? Using the conservation of energy, compute the explicit trajectory $\vec{f}_{\psi,\phi}(t)$ of the particle when

$$\frac{\vec{L}}{\|\vec{L}\|} = \big(\cos(\psi)\cosh(\phi), \sin(\psi)\cosh(\phi), -\sinh(\phi)\big)$$

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Note, the images of the trajectories will be contained in the unit disk $x^2 + y^2 < 1$. With this geometry, the disk is called the Poincaré disk.