Quantum States Localized on Lagrangian Submanifolds*

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(L, ω): Kostant-Souriau line bundle over symplectic manifold (X, ω).

**Definition (Souriau 1990)**

A *quantum state* is a state $m$ of $\text{Aut}(L)$

*State* of a group $G$: function $m : G \to \mathbb{C}$ such that

1. $m(e) = 1$,
2. the sesquilinear form

$$ (c, d)_m := \sum_{g, h \in G} \overline{c}_g d_h m(g^{-1} h) $$

on $\mathbb{C}[G] = \{\text{functions } G \to \mathbb{C} \text{ with finite support}\}$, is positive.
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$$ (c, d)_m := \sum_{g, h \in G} \bar{c}_g d_h m(g^{-1} h) \gg 0. $$

Gives rise to unitary $G$-module $\text{GNS}_m \ni \varphi$ such that $m(g) = (\varphi, g\varphi)$. 
(Put $(\cdot, \cdot)_m$ on $\mathbb{C}[G]$, divide out null vectors and complete; $\varphi = [\delta^e]$.)

**Localized Quantum States**

1. Quantum states
2. Localized states
3. Nilpotent groups
4. Compact groups
5. Euclid’s group
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5. Euclid's group

Quantum states

\((L, \varpi)\): Kostant-Souriau line bundle over symplectic manifold \((X, \omega)\).

**Definition (Souriau 1990)**

A *quantum state* (of \(\text{Aut}(L)\), for \(X\)) is a state \(m\) of \(\text{Aut}(L)\) such that

\[
\left| \sum_{j=1}^{n} c_j m(\exp(Z_j)) \right| \leq \sup_{x \in X} \left| \sum_{j=1}^{n} c_j e^{iH_j(x)} \right|
\]

for all choices of \(n \in \mathbb{N}\), \(c_j \in \mathbb{C}\) and *complete, commuting* \(Z_j \in \text{aut}(L)\) with hamiltonians \(H_j: \ H_j(x) = \varpi(Z_j(\xi))\).

- A *quantum representation* (of \(\text{Aut}(L)\), for \(X\)) is a unitary \(\text{Aut}(L)\)-module \(\mathcal{H}\) s.t. \(m(g) = (\varphi, g\varphi)\) is quantum \(\forall\) unit \(\varphi \in \mathcal{H}\).

- **Theorem** (Souriau). \(m\) quantum \(\Rightarrow\) \(\text{GNS}_m\) quantum.
(L, \wp): Kostant-Souriau line bundle over symplectic manifold (X, \omega).

**Definition (Souriau 1990)**

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for all choices of \( n \in \mathbb{N}, \ c_j \in \mathbb{C} \) and complete, commuting \( Z_j \in \text{aut}(L) \) with hamiltonians \( H_j: \ H_j(x) = \wp(Z_j(\xi)) \).

**Examples**

None. (Unless X is zero-dimensional.)

**Remark.** X is a coadjoint orbit of Aut(L). We might more modestly ask for states and representations of smaller groups (of which X is a coadjoint orbit).
X: coadjoint orbit of a connected Lie group G.

**Definition (Souriau 1990)**

A *quantum state* (of G, for X) is a state $m$ of G such that

$$\left| \sum_{j=1}^{n} c_j m(\exp(Z_j)) \right| \leq \sup_{x \in X} \left| \sum_{j=1}^{n} c_j e^{i\langle x, Z_j \rangle} \right|$$

for all choices of $n \in \mathbb{N}$, $c_j \in \mathbb{C}$ and commuting $Z_j \in \mathfrak{g}$.

**Examples**

Too many. (Unless X is zero-dimensional.)

- If $X = \{x\}$ is an integral point-orbit, then the unique quantum state for X is the character $m(\exp(Z)) = e^{i\langle x, Z \rangle}$. 
The statistical interpretation

Let \( \hat{g} := \) (compact) character group of the discrete additive group \( g \). We have a dense inclusion \( g^* \hookrightarrow \hat{g}, x \mapsto e^{i\langle x, \cdot \rangle} \), and projections

**Theorem**

*A state \( m \) of \( G \) is quantum for \( X \) \( \iff \) for each abelian \( a \subset g \), the state \( m \circ \exp|_a \) of \( a \) has its spectral measure concentrated on \( bX|_a \), the projection (in \( \hat{a} \)) of the closure \( bX \) of \( X \) (in \( \hat{g} \)).

This *spectral measure* is the probability measure \( \mu \) on \( \hat{a} \) such that \( (m \circ \exp|_a)(Z) = \int_{\hat{a}} \chi(Z) d\mu(\chi) \). (Bochner.)
Why “too many” quantum representations?

Because this (‘Bohr’) closure operation $b$ is drastic:

**Theorem (Howe-Z., dx.doi.org/10.1017/etds.2013.73)**

(a) If $G$ is noncompact simple, every nonzero coadjoint orbit is Bohr dense in $\hat{\mathfrak{g}}$, i.e. $bX = \hat{\mathfrak{g}}$.

(b) If $G$ is connected nilpotent, every coadjoint orbit is Bohr dense in its affine hull.

**Corollary**

(a) If $G$ is noncompact simple, every unitary representation of $G$ is quantum for every nonzero coadjoint orbit (!)

(b) If $G$ is connected nilpotent and $X$ spans $\mathfrak{g}^*$ (reduce to this case by dividing out $\text{ann}(X)$), a unitary representation of $G$ is quantum for $X \Leftrightarrow$ the center acts in it by the character $\exp(Z) \mapsto e^{i\langle X,Z \rangle}$.
So Souriau’s definition is not restrictive enough. 3 ways to proceed:

1. Hope that the much-needed selection will arise by restricting attention to states that extend to the whole Aut(L).

2. Suppress the Bohr closure implicit in the definition. For results along this line see arxiv.org/abs/1011.5056.

3. Take this closure seriously, because it allows interesting states:

Definition

Let $H \subset G$ be a closed subgroup and $Y \subset X_{|\mathfrak{h}}$ a coadjoint orbit of $H$. A quantum state $m$ for $X$ is **localized at** $Y \subset \mathfrak{h}^*$ if the restriction $m|_{\mathfrak{h}}$ is a quantum state for $Y$.

We also say that the state is **localized on** $\pi^{-1}(Y)$, where $\pi$ is the projection $X \to \mathfrak{h}^*$. One knows this set is generically a *coisotropic submanifold* — hence at least half-dimensional, and suitable for localizing a system on. We’ll mostly consider $Y = \{\text{pt}\}$. 

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Let $H \subset G$ be a closed subgroup and $Y \subset X_{\mathfrak{h}}$ a coadjoint orbit of $H$. A quantum state $m$ for $X$ is \textit{localized at} $Y \subset \mathfrak{h}^*$ if the restriction $m|_H$ is a quantum state for $Y$.

One should expect uniqueness of such a state when $\pi^{-1}(Y)$ is \textit{lagrangian} (half-dimensional): Weinstein (1982) called attaching state vectors to lagrangian submanifolds the \textsc{Fundamental Quantization Problem}.
G : connected, simply connected nilpotent Lie group,
X : coadjoint orbit of G,
\( x \) : chosen point in X.
A connected subgroup \( H \subset G \) is **subordinate to** \( x \) if, equivalently,

- \( \{ x_{|h} \} \) is a point-orbit of \( H \) in \( \mathfrak{h}^* \)
- \( \langle x, [\mathfrak{h}, \mathfrak{h}] \rangle = 0 \)
- \( e^{ix \circ \log}_{|H} \) is a character of \( H \).

**Theorem**

*Let \( H \subset G \) be maximal subordinate to \( x \in X \). Then there is a unique quantum state for \( X \) localized at \( \{ x_{|h} \} \subset \mathfrak{h}^* \), namely*

\[
m(g) = \begin{cases} 
e^{ix \circ \log}(g) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}
\]

*Moreover \( \text{GNS}_m = \text{ind}_H^G e^{ix \circ \log}_{|H} \) (discrete induction).*

\( \mathfrak{a} \subset \mathfrak{h} \Rightarrow x_{|\mathfrak{a}} \text{ certain; } \mathfrak{a} \cap \mathfrak{h} \Rightarrow x_{|\mathfrak{a}} \text{ equidistributed in } \hat{\mathfrak{a}}. \)
Nilpotent groups

Remark
Kirillov (1962) used $I(x, H) := \text{Ind}_H^G e^{ix \circ \log |H}$ (usual induction). This is

(a) irreducible $\iff$ $H$ is a polarization at $x$ (: subordinate subgroup such that the bound $\dim(G/H) \geq \frac{1}{2} \dim(X)$ is attained);

(b) equivalent to $I(x, H')$ if $H \neq H'$ are two polarizations at $x$.

In contrast:

Theorem

Let $H \subset G$ be subordinate to $x$. Then $i(x, H) := \text{ind}_H^G e^{ix \circ \log |H}$ is

(a) irreducible $\iff$ $H$ is maximal subordinate to $x$;

(b) inequivalent to $i(x, H')$ if $H \neq H'$ are two polarizations at $x$. 
Example: Extended Galilei group $G = \left\{ g = \begin{pmatrix} 1 & b & \frac{1}{2} b^2 & a \\ 1 & b & c & e \\ \end{pmatrix} \right\}$

B and C are maximal subordinate but only C is a polarization.
So $i(x, C)$, $I(x, C)$, $i(x, B)$ are irreducible but $I(x, B)$ is not.

All act by $(g\psi)(\begin{pmatrix} r \\ t \end{pmatrix}) = e^{-ia} e^{-i\{b(r-c) - \frac{1}{2} b^2(t-e)\}} \psi(\begin{pmatrix} r-c-b(t-e) \\ t-e \end{pmatrix})$, but

1. $I(x, B)$ in $L^2$ functions of $\begin{pmatrix} r \\ t \end{pmatrix}$
2. $I(x, C)$ in $L^2$ solutions of Schrödinger’s equation $i\partial_t \psi = \frac{1}{2} \partial^2_r \psi$
3. $i(x, C)$ in almost periodic solutions, norm $\lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} |\psi|^2 \, dr$
4. $i(x, B)$ in $\ell^2$ functions — no Schrödinger equation needed!
Compact groups

Theorem

*Every quantum representation of a compact Lie group $G$ is continuous. The irreducible with highest weight $\lambda$ is quantum for the coadjoint orbit with dominant element $\mu \iff \lambda \leq \mu$.*

So even for compact $G$, Souriau’s definition does not recover the usual ‘orbit method’ (which posits $\lambda = \mu$). In contrast we have, with $T \subset G$ a maximal torus:

**Theorem**

- If $\mu$ is dominant integral, then there is a unique quantum state $m$ for $X = G(\mu)$ localized at $\{\mu|_t\} \subset t^*$; $GNS_m$ is the irreducible representation with highest weight $\mu$.
- If $\mu$ is dominant and not integral, then there is no such state.
Euclid’s group $G = \left\{ g = \begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} : A \in \text{SO}(3), c \in \mathbb{R}^3 \right\}$

**Example: TS$^2$**

$G$ acts naturally and symplectically on the manifold $X \cong \text{TS}^2$ of oriented lines (a.k.a. light rays) in $\mathbb{R}^3$. 2-form $\omega$:

$$\omega = k \, d\langle u, \, dr \rangle + s \, \text{Area}_{S^2}.$$ 

The moment map

$$\Phi(u, r) = \begin{pmatrix} r \times ku + su \\ ku \end{pmatrix}$$

makes $X$ into a coadjoint orbit of $G$. 
Case $s = 0$:

We have localized states on 3 types of lagrangians:

1. The tangent space at the north pole
2. The zero section
3. The equator’s normal bundle

(a) $m \begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{i(k e_3, c)} & \text{if } A e_3 = e_3, \\
0 & \text{otherwise.} \end{cases}$

(b) $m \begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} = \frac{\sin \|k c\|}{\|k c\|}$

(c) $m \begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} = \begin{cases} J_0(\|k c_\perp\|) & \text{if } A e_3 = \pm e_3, \\
0 & \text{otherwise.} \end{cases}$

The resulting GNS modules can be realized as various spaces of solutions of Helmholtz’s equation $\Delta \psi + k^2 \psi = 0$, with $G$-action $(g \psi)(r) = \psi(A^{-1}(r - c))$. 
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Cyclic vector:

(a) $\psi(r) = e^{-ikz}$

(b) $\psi(r) = \frac{\sin \|k r\|}{\|k r\|}$

(c) $\psi(r) = J_0(\|k r_\perp\|)$
Case $s = 1$ (zero section no longer lagrangian):

The unique quantum state localized on the tangent space (a) becomes

$$m \begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{i\alpha}e^{i\langle ke_3, c \rangle} & \text{if } A = e^{j(\alpha e_3)}, \quad (j(\alpha) = \alpha \times \cdot) \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{GNS}_m = \{ \ell^2 \text{ sections } b \text{ of the tangent bundle } TS^2 \to S^2 \},$$

with G-action

$$(gb)(u) = e^{\langle u, kc \rangle J} Ab(A^{-1}u)$$

where $J\delta u = j(u)\delta u$. Putting

$$F(r) = (B + iE)(r) := \sum_{u \in S^2} e^{-\langle u, kr \rangle J} (b - iJb)(u)$$

one obtains a Hilbert space of almost-periodic solutions of the reduced Maxwell equations

$$\begin{cases} \text{div } B = 0, & \text{curl } B = kB, \\ \text{div } E = 0, & \text{curl } E = kE, \end{cases}$$

with G-action

$$(gF)(r) = AF(A^{-1}(r - c)).$$

The cyclic vector is $F(r) = e^{-ikz}(e_1 - ie_2)$. 