Averaging of Dirac structures

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Gone Fishing, Berkeley, 2014
Preliminaries: singular foliated structures

\( \Pi \text{ Poisson induces a symplectic foliation: leafwise } w = \{w_S\}_{S \in S}. \)

\( \Pi \text{ Dirac induces a presymplectic foliation: leafwise } w = \{w_S\}_{S \in S}. \)

Smoothness: for each \( f \in C^\infty(M) \) and \( m \in M \),
\( i_X m w_S = -d_m f \) defines a smooth vector field.

For \( X = X_f, Y = Y_f \) such vector fields,
\( \Pi(d_f, d_g) = w(X_f, X_g) \) (on each \( S \subset M \), \( w = w_S \)).

Reciprocally, given \( \Pi \text{ Poisson we can define } w \) by this formula (\( X_f \) is constructed just in terms of \( \Pi \)).

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\( D_w m = \{(X, \alpha) \in T_m M: X \in T_m S, \alpha|_{T_m S} = -i_X w_m\} \)

Then \( w = \{w_S\}_{S \in F} \) is smooth iff \( D_w \) is a smooth (Dirac) subbundle.

Reciprocally, given \( D \), we have \( \text{pr}(D) \subset TM \) integrable, and the presymplectic leaves are its maximal integral submanifolds. On \( S \), a leafwise presymplectic structure is \( w_m(X, Y) = -\alpha(Y) \).
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Take \( \nu \in \Omega^2(M) \). It induces a distribution \( D^\nu \) on \( TM \):

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D^\nu = \text{Graph}(\nu) = \{(X, i_X \nu) : X \in \Gamma TM\} \subset TM.
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We have:

\( D^\nu \) is Dirac iff \( \nu \) is closed.

There is a contravariant version: any \( \Pi \in \Gamma \Lambda^2 TM \) induces

\[ D^\Pi = \text{Graph}(\Pi) = \{(\iota_\alpha \Pi, \alpha) : \alpha \in \Gamma T^* M\} \subset TM. \]

Observe that \( \text{pr}(D^\Pi) = \Pi^\#(T^* M) \). The corresponding criterion is

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Gauge transformations

Consider a Dirac manifold $(M, D)$. Let $B \in \Omega^2(M)$. Then, to $D$ and $B$ we can associate a new distribution

$$\tau_B(D) = \{(X, \alpha - i_X B) : (X, \alpha) \in \Gamma(D)\}. $$

This new $\tau_B(D)$ is Dirac iff $B$ is closed. In this case, $B$ is called a gauge transformation (Bursztyn and Radko, ˇSevera and Weinstein). It is an exact gauge transformation if $B$ is exact. One of the main purposes of our work is to study the role of these gauge transformations in the averaging theory of Poisson and Dirac structures under a certain class of actions of compact groups.
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Compatible actions

Let $M$ be a manifold and $G$ a connected compact Lie group. Denote by $a_M \in \Gamma TM$ the infinitesimal generator of the action $\Phi : G \times M \to M$, associated to an element $a \in \mathfrak{g}$,

$$a_M(m) = \left. \frac{d}{dt} \right|_{t=0} \Phi_{\exp(ta)}(m).$$

For any $F \in \mathcal{T}_s^r(M)$, its $G$–average is

$$\langle F \rangle^G = \int_G \Phi_{\gamma}^* F \, dg.$$

We also have another averaging $\delta^G : \text{Hom}_\mathbb{R}(\mathfrak{g}; \mathcal{T}_s^r(M)) \to \mathcal{T}_s^r(M)$,

$$\delta^G(\lambda) = \int_\mathcal{D} \left( \int_0^1 \Phi_{\exp(ta)}^* \lambda_a \, dt \right) \exp^* \, dg,$$

(where $\exp|_\mathcal{D} : \mathcal{D} \to G \setminus C(e)$ is a diffeomorphism).
Now let \((M, D)\) be a Dirac manifold, \((S, w)\) the associated presymplectic foliation. Suppose \(G\) acts on \(M\) preserving each leaf \(S\) of \(S = \text{pr}(D)\), but the action not necessarily canonical with respect to \(D\).
The action is \textit{compatible} if there exists a morphism \(\rho \in \text{Hom}_{\mathbb{R}}(g; \Omega^1(M))\) such that

\[ i_{a_M} w_S = -i^*_S \rho_a , \]

for all \(a \in g\), where \(i_S : S \hookrightarrow M\) is the canonical injection.
Equivalently,

\[ (a_M, \rho_a) \in \Gamma(D) , \]

for all \(a \in g\).
Main results
Regarding averaging of Dirac structures:

**Theorem 1**

If the $G$–action is compatible on $(M, D)$, then the average $\langle \omega \rangle^G$ is smooth, and can be represented as

$$\langle \omega \rangle^G = \omega - i_s^\ast d\Theta,$$

where $\Theta \in \Omega^1(M)$ is the 1–form given by

$$\Theta := \delta^G(\rho).$$

The associated Dirac structure $\overline{D} := D\langle \omega \rangle^G$ is $G$–invariant and related to $D$ by an exact gauge transformation, $B = d\Theta$

$$\overline{D} = \{(X, \alpha - i_X d\Theta) : (X, \alpha) \in D\}.$$
For Poisson structures around (singular) symplectic leaves:

**Theorem 2**

Let $S$ be a symplectic leaf of the foliation induced by $\Pi$. Assume that the action of $G$ on $M$ is compatible with $\Pi$ ($a_M = \Pi^\# \rho_a$ for all $a \in \mathfrak{g}$).

1. If $G$ acts canonically on $S$ (that is, $\iota^*_S \rho_a$ is closed on $S$), then, it determines a $G$—invariant Poisson bi-vector $\overline{\Pi}$, well-defined in a $G$—invariant neighborhood $N$ of $S$ in $M$. Both structures are related by an exact gauge transformation:

$$\overline{\Pi}^\# = \Pi^\# \circ (\text{Id} + (d\Theta)^\# \circ \Pi^\#)^{-1}.$$ 

2. Moreover, if $S \subset M^G$ (the set of fixed points), then, the germs at $S$ of $\overline{\Pi}$ and $\Pi$ are isomorphic by a local Poisson diffeomorphism $\phi : N \to M$,

$$\phi^* \Pi = \overline{\Pi}, \quad \phi|_S = \text{id}$$
The proof of this results uses the exact gauge form from Theorem 1, $\Theta$, and a version of Moser path method adapted to contravariant tensors.

If along the paths $\phi_t$ the structures are related by exact gauge transformations, one can check that the time-dependent vector field $Z_t$ given by

$$Z_t = -\Pi^\#_t(\Theta) = -\Pi^\# \circ (\text{Id} + t(d\Theta)^\# \circ \Pi^\#)^{-1}(\Theta),$$

satisfies the homotopy equation in terms of the Schouten-Nijenhuis bracket (Crainic, Frejlich, Mărcuț, Vorobiev)

$$[Z_t, \Pi_t] = -\frac{d\Pi_t}{dt}.$$ 

The sought-after Poisson diffeomorphism is then given by the flow at time 1, $\phi = \text{Fl}^{Z_t}_{Z_t}\big|_{t=1}$. 
Next, we want to consider coupling Dirac structures. Recall that a set of geometric data on a foliated manifold consist of a normal bundle $\mathbb{H}$ of $\mathcal{F}$ (equivalently, a connection $\gamma$), a horizontal (coupling) 2–form $\sigma \in \Gamma \Lambda^2(\mathbb{V}^0)$ on $M$, and a leaf-tangent Poisson bivector $P \in \Gamma \Lambda^2(\mathbb{V})$. There is a one-to-one correspondence between integrable geometric data and coupling Dirac structures $D \subset \mathbb{T}M$, i.e., such that the tangent distribution

$$\mathbb{H}_m = \{ Z \in T_m M : \exists \alpha \in \mathbb{V}^0 \text{ such that } (Z, \alpha) \in D \},$$

is a normal bundle of $\mathcal{F}$ (Dufour and Wade, Vaisman).
Let:

- \((M, \mathcal{F}, P)\) be a vertical Poisson structure (\(\mathcal{F}\) regular Poisson foliation).

- \(G\) be a connected compact Lie group with a locally Hamiltonian action \(a_M = P^\#(\mu_a)\) for all \(a \in \mathfrak{g}\), with \(\mu_a\) closed).

- \(D\) be a coupling Dirac structure with geometric data \((\gamma, \sigma, P)\).

Then, there exists a \(G\)–averaged \(\overline{D}\) which is a \(G\)–invariant coupling Dirac structure, that is, its geometric data are \((\overline{\gamma}, \overline{\sigma}, P)\), which can be explicitly computed from \(\gamma, \sigma\) and \(\mu\).
Obstructions to the existence of Hamiltonian actions

Instead of a foliation we now consider Poisson fibrations $(\pi : M \to B, P)$ (Brahic, Fernandes, Vaisman). We assume:

- A Hamiltonian action with momentum map $J \in \text{Hom}(g; \mathcal{C}^\infty(M))$, so $a_M = P^# dJ_a$ for each $a \in g$.
- A coupling Dirac structure $D$, with associated data $(\gamma, \sigma, P)$.

In general, there are obstructions of cohomological type, which lead to the so-called adiabatic hypothesis in the context of geometric phases on Poisson fiber bundles (Marsden, Montgomery, Ratiu).
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In general, the action is not Hamiltonian with respect to \(D\), but as there exists a \(G\)–invariant \(\overline{D}\), and the action is already assumed to be Hamiltonian with respect to \(P\), a natural question arises: when is the action Hamiltonian with respect to \(\overline{D}\)?
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The $G$—action is Hamiltonian on the Dirac manifold $(M, \overline{D})$ with momentum map $J$, that is,

$$(a_M, dJ_a) \in \Gamma(\overline{D}), \quad \forall a \in \mathfrak{g},$$

if and only if

$$\left\langle d_{1,0}^\gamma J \right\rangle^G = 0.$$
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Notice that this condition is expressed in terms of the momentum map $J$ (which reflects the action of $G$), and the connection $\gamma$ (which comes from the geometric data).