Toric log symplectic manifolds

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Outline

Review of Delzant classification

Toric log symplectic manifold
  Log affine manifolds

Tropical domains and welding

Classification and examples

Reference
Toric symplectic manifolds

A toric symplectic manifold is a triple \((M, \omega, T^n)\) such that

- \(M\) is a compact 2n-manifold;
- \(\omega \in \Omega^2(M)\) is a symplectic structure;
- there is an effective action on \((M, \omega)\) by \(T^n = (S^1)^n\).
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- there is an effective action on $(M, \omega)$ by $T^n = (S^1)^n$.

We say $(M, \omega, T^n)$ is Lagrangian, if for $a, b \in t$,

$$\omega(\rho(a), \rho(b)) = 0.$$
**Hamiltonian action**

We say \((M, \omega, T^n)\) is **Hamiltonian**, if there exists a momentum map \(\mu : M \to t^*\) such that for every \(a \in t\),

\[
i_{\rho(a)}\omega = d\mu_a.
\]
Hamiltonian action

We say \((M, \omega, T^n)\) is **Hamiltonian**, if there exists a momentum map \(\mu : M \to t^*\) such that for every \(a \in t\),

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\iota_{\rho(a)} \omega = d\mu_a.
\]

If we write the momentum image as \(\Delta = \mu(M)\), then equivalently, \((M, \omega, T^n)\) is Hamiltonian if \(\iota_{\rho} \omega\) descends to \(\xi \in \Omega^1(\Delta) \otimes t^*\) which satisfies

\[
[\xi] = 0 \in H^1(\Delta) \otimes t^*.
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That is, \(\iota_{\rho} \omega\) descends to an exact \(t^*\)-valued 1-form \(\xi\) on \(\Delta\).
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\((M, \omega, T^n)\) is Hamiltonian \(\implies\) \((M, \omega, T^n)\) is Lagrangian.
Delzant polytopes

If we treat $t^*$ as an affine space, then its (co)tangent bundle is trivialized.

\[ T t^* = t^* \times t^*, \quad T^* t^* = t^* \times t \]
Delzant polytopes

If we treat $\mathfrak{t}^*$ as an affine space, then its (co)tangent bundle is trivialized.

$$ T\mathfrak{t}^* = \mathfrak{t}^* \times \mathfrak{t}^* , \quad T^*\mathfrak{t}^* = \mathfrak{t}^* \times \mathfrak{t} $$

A Delzant polytope is a convex polytope $\Delta \subset \mathfrak{t}^*$ such that

- A face $f \subset \Delta$ has a rational normal vector $v_f$;
- At each vertex $v$, the vectors $v_f$ of the faces meeting $v$ spans $\mathfrak{t}_\mathbb{Z}^*$. 
**Delzant polytopes**

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**Delzant Theorem**

There is a one-to-one bijection between Hamiltonian toric symplectic manifolds \((M, \omega, T^n, \mu)\) and the Delzant polytopes \( \Delta \).
\[ \Delta \sim (M, \omega, T^n) \]

Given a Delzant polytope \( \Delta \subset t^* \), the natural symplectic structure on the cotangent bundle

\[ T^* \Delta = \Delta \times t \]

is invariant under the \( t_\mathbb{Z} \) translation on the fibers. We denote the quotient by

\[ (\tilde{M} = T^* \Delta / t_\mathbb{Z}, \tilde{\omega}) \]

which has a residual \( T^n \) action.
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By performing the symplectic cuts along the boundary components of $(\tilde{M}, \tilde{\omega}, T^n)$, we obtain a compact Hamiltonian toric symplectic manifold $(M, \omega, T^n)$. 


A **toric log symplectic manifold** is quadruple \((M, Z, \omega, T^n)\) such that

- \(M\) is a compact \(2n\)-manifold;
- \(Z = \bigcup Z_i\) is a collection of normal crossing hypersurfaces;
- \(\omega \in \Omega^2(M, \log Z)\) is a log symplectic structure;
- there is an effective action on \((M, \omega)\) by \(T^n\).
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As before, we say \((M, Z, \omega, T^n)\) is **Lagrangian**, if for \(a, b \in t\),

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The notion of Hamiltonian is a bit tricky, so we need define the notion of log affine manifolds.
**Log affine manifolds**

A **log affine manifold** is a triple \((\Delta, D, \xi)\) such that

- \(\Delta\) is a n-manifold with corners;
- \(D = \bigcup D_i\) is a collection of normal crossing hypersurfaces with \(\partial \Delta \subset D\);
- \(\xi \in \Omega^1(X, \log D) \otimes t^*\) is a \(t^*\)-valued log 1-form trivializing the log tangent bundle \(T(X, - \log D)\).

\[
\xi \in \Omega^1(X, \log D) \otimes t^* \iff T(X, - \log D) \xrightarrow{\xi} X \times t^*
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$$\xi \in \Omega^1(X, \log D) \otimes t^* \iff T(X, -\log D) \xrightarrow{\xi} X \times t^*$$

Proposition (Gualtieri-L-Pelayo-Ratiu)

If $(M, Z, \omega, T^n)$ is Lagrangian toric, then

$$(\Delta = M/T^n, D = Z/T^n, \xi = \iota_\rho \omega)$$

is log affine. ($\iota_\rho \omega$ is basic and descends $\xi$.)
Hamiltonian action

The logarithmic cohomology of \((\Delta, D)\) decomposes as follows:

\[ H^n(M, \log Z) = H^n(M) \oplus \left( \bigoplus_i H^{n-1}(Z_i) \right) \oplus \left( \bigoplus_{i>j} H^{n-2}(Z_i \cap Z_j) \right) \oplus \ldots \]
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\]

A Lagrangian toric $(M, Z, \omega, T^n)$ is **Hamiltonian** if

\[
i_\rho \omega \in \Omega^1(M, \log Z) \otimes t^*
\]

has trivial class in $H^1(M) \otimes t^*$. This is equivalent to say that $i_\rho \omega$ descends to

\[
\xi \in \Omega^1(\Delta, \log D) \otimes t^*
\]

which has trivial class in $H^1(\Delta) \otimes t^*$, i.e. $(\Delta, D, \xi)$ has trivial **affine monodromy**.
Goal: Classify the Hamiltonian \((M, Z, \omega, T^n)\) using \((\Delta, D, \xi)\).
Tropical domains

**Idea:** Think $t^*$ as an affine space, and compatifiy $t^*$ to a log affine space, which is called a tropical domain.
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Example 1. Compactifying $\mathbb{R}$ at $\pm \infty$, we obtain a tropical domain isomorphic to the closed interval $[0, 1]$ with a log 1-form

$$\xi = \left( \frac{a}{t} + \frac{b}{t-1} \right) dt$$
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**Example 2a.** Compactifying $\mathbb{R}^2$ in the (reserve) directions of $(1, 0), (0, 1), (-1, -1)$ we obtain a tropical domain isomorphic to the triangle

$$X = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1\}$$

with the log 1-form $\xi \in \Omega^1(X, \partial X) \otimes \mathbb{R}^2$ such that the residues of $\xi$ are exactly $(1, 0), (0, 1), \text{ and } (-1, -1)$. 
Tropical welding

In general, we may pick any fan $\Sigma$ of $t^*$ and partially compactify $t^*$ in the (reverse) directions of the vectors generating its 1-dim cones to obtain an tropical domain $X_\Sigma$. 
Tropical welding

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For two such tropical domains $(X, \xi)$ and $(X', \xi')$, we may weld $X$ and $X'$ along the boundary components $D_i$ and $D'_j$ if

$$\text{Res}_i \xi = \text{Res}_j \xi' \in t^*$$

and all adjacent residues also match. The result of such welding is again a log affine manifold, which is called a tropical welded space.
Tropical welding

In general, we may pick any fan $\Sigma$ of $t^*$ and partially compactify $t^*$ in the (reverse) directions of the vectors generating its 1-dim cones to obtain an **tropical domain** $X_\Sigma$.

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and all adjacent residues also match. The result of such welding is again a log affine manifold, which is called a **tropical welded space**.

**Example 2b.** We may glue 8 copies of the triangles in Example 2a to obtain a log affine manifold isomorphic to $S^2$ with $\xi \in \Omega^1(S^2, \log D) \otimes \mathbb{R}^2$ where $D$ is 3 transversally intersecting great circles.
Log affine polytopes

Let \((X, D, \xi)\) be a tropical welded space. We say \(\Delta \subset X\) is a **Dezllant log affine polytope** if \(\Delta\) is compact and intersects the interior of each tropical domain in a (possibly non-compact) Delzant polytope.
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**Proposition (Gualtieri-L-Pelayo-Ratiu)**

If \((M, Z, \omega, T^n)\) is Hamiltonian toric, then

\[
(\Delta = M/ T^n, D = Z/ T^n, \xi = \iota_\rho \omega)
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is a Delzant log affine polytope.
Chern class

Let \((M, Z, \omega, \xi)\) be a Lagrangian toric log symplectic manifold. In particular, the \(T^n\) action is locally free. There is a reverse operation to symplectic cut, which we called symplectic uncut, which renders the \(T^n\) action free.
Chern class

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Let \(M'\) be the resulting uncut manifold. Then \(M'\) is a \(T^n\) principal bundle over \(\Delta\). The n-tuple Chern class of \((M, Z, \omega, T^n)\) is

\[ c_1(M) = c_1(M') = (c^1_1, \ldots, c^n_1) \in H^2(\Delta) \otimes t \]
Classification and examples

Chern class

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Lemma (Gualtieri-L-Pelayo-Ratiu)

For a Lagrangian toric log symplectic manifold \((M, Z, \omega, \xi)\),

\[
c_1(M) \cdot \xi = \sum_{j=1}^{n} c_1^j \wedge [\xi_j] = 0 \in H^3(\Delta, \log D).
\]
Delzant correspondence

**Thereom (Gualtieri-L-Pelayo-Ratiu)**

The classification of Hamiltonian log symplectic manifolds are as follows:

\[(M, Z, \omega, T^n)\] Hamiltonian log symplectic manifold
\[\iff\] \[(\Delta, D, \xi)\] Delzant log affine polytope
\[c_1 \in \Omega^2(\Delta) \otimes t\text{ such that } c_1(M) \cdot \xi = 0\]
\[\text{the affine space } H^2(\Delta, \log D)\]
Examples

**Example 2c.** In Example 2c, we have a log affine manifold \((S^2, D, \xi)\) where \(D\) is 3 transversally intersecting great circles. If we take \(c_1 = 0\), then \(c_1 \cdot \xi = 0\) holds. This also implies

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M = S^2 \times T^2, \quad Z = D \times T^2.
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M = S^2 \times T^2, \quad Z = D \times T^2.
\]

Since \(T(T^2) = T^2 \times t\), we have a natural \(t\)-valued 1-form \(\theta = (\theta^1, \ldots, \theta^n) \in \Omega^1(T^2) \otimes t\). For any \(\beta \in H^2(S^2, \log D)\), we take

\[
\omega = \sum_{j=1}^{n} \theta^j \wedge \xi_j + \beta \in \Omega^2(M, \log Z).
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\[ \omega = \sum_{j=1}^{n} \theta^j \wedge \xi_j + \beta \in \Omega^2(M, \log Z). \]

Then \((M, Z, \omega, T^2)\) is a Hamiltonian toric log symplectic manifold.

Note: \(H^2(S^2, \log D) = \mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^6 = \mathbb{R}^{10}\) and we have a 10-dim moduli space of Hamiltonian toric log symplectic structures on \((M, Z, T^2)\).
Examples

Example 2c. cont... Another way to think about Example 2c is as follows. Because $\xi$ trivializes $T(S^2, -\log D)$, so $\xi$ also trivializes $T^*(S^2, \log D)$, i.e.

$$T^*(S^2, \log D) \cong S^2 \times t$$
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Let $\pi_0$ be the natural Poisson structure on $T^*(S^2, - \log D)$. Then

$$M = T^*(S^2, \log D)/t_{\mathbb{Z}}$$

and $\pi_0$ descends to a Poisson structure on $M$ whose inverse is

$$\omega_0 = \sum_{j=1}^{n} \theta^j \wedge \xi_j.$$
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$$\omega_0 = \sum_{j=1}^{n} \theta^j \wedge \xi_j.$$ 

Remark: There is no distinguished $\omega_0$ if the chern class $c_1 \neq 0$. 
Examples

Example 3. We compactify $\mathbb{R}^2$ to a hexagon tropical domain. Welding four such hexagon tropical domains give us a surface of genus 2.
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Half of the surface, i.e. a torus with a disk removed, is a log affine polytope. Taking the Chern class $c_1 = 0$, we obtain a 4-dim Hamiltonian log symplectic manifold $(M, Z, \omega, T^2)$ such that

$$M = S^1 \times (((S^1 \times S^2) \# (S^1 \times S^2)))$$
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Interestingly, $M$ has vanishing Sieberg-Witten invariant, so $M$ is not symplectic, but $M$ is log symplectic.
References


M. Gualtieri, S. Li, A. Pelayo and T. Ratiu, Tropical moment maps for toric log symplectic manifolds, arXiv:1407.3300

Thank you!