Cluster Structures on Drinfeld Doubles

M. Gekhtman (joint with M. Shapiro and A. Vainshtein)

Gone Fishing 2014
Cluster Algebras

A seed (of geometric type) - a pair \( \Sigma = (\tilde{x}, \tilde{B}) \):

- extended cluster \( \tilde{x} = (x_1, \ldots, x_n, \ldots, x_{n+m}) \)
- stable extended exchange matrix \( \tilde{B} \) - an \( n \times (n+m) \) integer matrix whose \( n \times n \) principal part \( B \) is skew-symmetrizable.

(Skew-symmetric case: \( \tilde{B} \) is an adjacency matrix of a quiver \( Q \).)

The adjacent cluster in direction \( k \in [1, n] \):

\[ x_k = (x_{\{x_k\}} \cup \{x'_k\}), \]

where the new cluster variable \( x'_k \) is given by the exchange relation

\[ x_k x'_k = \prod_{1 \leq i \leq n+m, b_{ki} > 0} x_i^{b_{ki}} \prod_{1 \leq i \leq n+m, b_{ki} < 0} x_i^{-b_{ki}}; \]

\( \tilde{B}' \) is obtained from \( \tilde{B} \) by a matrix mutation in direction \( k \):

\[ b'_{ij} = \begin{cases} 
- b_{ij}, & \text{if } i = k \text{ or } j = k \\
& \text{otherwise.}
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$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k; \\ b_{ij} + |b_{ik}| b_{kj} + |b_{ik}| b_{kj}^2, & \text{otherwise.} \end{cases}$$
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Cluster structure \( C(\tilde{B}) \): The set of all seeds mutation equivalent to \( \Sigma \).

Cluster algebra (of geometric type) \( \mathcal{A} = \mathcal{A}(\tilde{B}) \) is generated by all cluster variables in all seeds mutation equivalent to \( \Sigma \).

Upper cluster algebra \( \overline{\mathcal{A}} = \overline{\mathcal{A}(C)} = \overline{\mathcal{A}(\tilde{B})} \) is the intersection of the rings of Laurent polynomials in cluster variables taken over all seeds in \( C(\tilde{B}) \).
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\[ \mathcal{A}(\mathcal{C}) \subseteq \overline{\mathcal{A}(\mathcal{C})} \]
Compatible Poisson Brackets

A Poisson bracket \( \{ \cdot, \cdot \} \) on \( F = C(x_1, \ldots, x_n + m) \) is compatible with the cluster algebra \( A \) if, for any extended cluster \( \tilde{x} = (x_1, \ldots, x_n + m) \)

\[
\{ x_i, x_j \} = \omega_{ij} x_i x_j,
\]

where \( \omega_{ij} \in \mathbb{Z} \) are constants for all \( i, j \in [1, n + m] \).

Theorem (G.-S.-V.) Assume that \( \tilde{B} \) is of full rank. Then there is a Poisson bracket compatible with \( A(\tilde{B}) \).
A Poisson bracket \( \{ \cdot, \cdot \} \) on \( \mathcal{F}_\mathbb{C} = \mathbb{C}(x_1, \ldots, x_{n+m}) \) is \textit{compatible} with the cluster algebra \( \mathcal{A} \) if, for any extended cluster \( \tilde{x} = (x_1, \ldots, x_{n+m}) \)

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Theorem (G.-S.-V.)

Assume that $\tilde{B}$ is of full rank. Then there is a Poisson bracket compatible with $\mathcal{A}(\tilde{B})$. 
Global Toric Action

Local toric action:

$$ T_d^W(x_i) = x_i \prod_{\alpha=1}^{r} d_\alpha^{w_i\alpha}, \quad i \in [n+m], \quad d = (d_1, \ldots, d_r) \in (\mathbb{C}^*)^r, $$

where $W = (w_{i\alpha})$ is an integer $(n + m) \times r$ weight matrix of full rank.
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Local toric action:

\[ \mathcal{T}_d^W(x_i) = x_i \prod_{\alpha=1}^{r} d_i^{w_{i\alpha}}, \quad i \in [n + m], \quad \mathbf{d} = (d_1, \ldots, d_r) \in (\mathbb{C}^*)^r, \]

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Compatibility condition:

\[ \begin{align*}
\mathcal{F}_C = \mathbb{C}(\tilde{x}) & \quad \longrightarrow \quad \mathcal{F}_C = \mathbb{C}(\tilde{x}') \\
\mathcal{T}_d^W & \quad \longrightarrow \quad \mathcal{T}_d^{W'}
\end{align*} \]
Global Toric Action

Local toric action:

$$\mathcal{T}_d^W(x_i) = x_i \prod_{\alpha=1}^{r} d_{\alpha}^{w_{i\alpha}}, \quad i \in [n + m], \quad d = (d_1, \ldots, d_r) \in (\mathbb{C}^*)^r,$$

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Compatibility condition:

$$\mathcal{F}_C = \mathbb{C}(\tilde{x}) \xrightarrow{\mathcal{T}_d^W} \mathcal{F}_C = \mathbb{C}(\tilde{x}')$$

If local toric actions at all clusters are compatible, they define a global toric action $\mathcal{T}_d$ on $\mathcal{F}_C$. 
Key Observation

Let \((V, \{\cdot, \cdot\})\) be a Poisson variety that

- possesses a coordinate system \(\tilde{x} = (x_1, \ldots, x_{n+m})\) with Poisson relations as above (log-canonical) for some \(\omega_{ij} \in \mathbb{Z}\);
- admits an action of \((\mathbb{C}^*)^m\) that induces a local toric action of rank \(m\) on \(\tilde{x}\).

Then there exists a unique skew-symmetric cluster structure \(\mathcal{C}(\tilde{B})\) with the initial extended cluster \(\tilde{x}\) and stable variables \(x_{n+1}, \ldots, x_{n+m}\) that is compatible with \(\{\cdot, \cdot\}\) and such that the local toric action above extends to a global toric action.
\((G, \{\cdot, \cdot\})\) is called a **Poisson–Lie group** if the multiplication map

\[ G \times G \ni (x, y) \mapsto xy \in G \]

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We are interested in the case

- G is a simple complex Lie group;
- \{·, ·\} = \{·, ·\}_r is associated with a classical R-matrix r - a solution of the CYBE:
A Poisson–Lie group $(\mathcal{G}, \{\cdot, \cdot\})$ is called a Poisson–Lie group if the multiplication map

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- $\mathcal{G}$ is a simple complex Lie group;
- $\{\cdot, \cdot\} = \{\cdot, \cdot\}_r$ is associated with a classical R-matrix $r$ - a solution of the CYBE:

$$\{X \otimes X\}_r := [r, X \otimes X]$$
Belavin-Drinfeld Classification

Up to an automorphism, every classical R-matrix $r$ belongs to one of disjoint classes $\mathcal{R}_T$ specified by the Belavin-Drinfeld data

$$T = (\Gamma_1, \Gamma_2, \tau), \quad (\Gamma_{1,2} \subset \Delta, \ \tau: \Gamma_1 \to \Gamma_2),$$
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$$T = (\Gamma_1, \Gamma_2, \tau), \quad (\Gamma_1, \Gamma_2 \subset \Delta, \tau : \Gamma_1 \to \Gamma_2),$$

where $\Delta$ is the set of simple positive roots and $\tau$ is an isometry s.t.

$$\forall \alpha \in \Gamma_1 \exists m \in \mathbb{N} : \tau^j(\alpha) \in \Gamma_1 \ (j = 0, \ldots, m - 1), \ \tau^m(\alpha) \notin \Gamma_1.$$
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$\mathcal{R}_T$ is linear space of dimension $\frac{k_T(k_T - 1)}{2}$, where $k_T = \dim \mathfrak{h}_T$,

$$\mathfrak{h}_T = \{ h \in \mathfrak{h} : \alpha(h) = \beta(h) \text{ if } \beta = \tau^j(\alpha) \},$$
Let $G$ be a simple complex Lie group. For any Belavin-Drinfeld triple $T = (\Gamma_1, \Gamma_2, \tau)$ there exists a cluster structure $\mathcal{C}_T$ on $G$ such that

- The number of stable variables is $2 \dim h_T$.
- The corresponding extended exchange matrix has a full rank.
- $\mathcal{C}_T$ is regular, and the corresponding upper cluster algebra $A_{\mathcal{C}_T}(G)$ is naturally isomorphic to $O(G)$.
- The global toric action is generated by the action of $\exp(h_T) \times \exp(h_T)$ on $G$ given by $(H_1, H_2)(X) = H_1 X H_2$.
- For any $r \in \mathbb{R} T$, $\{\cdot, \cdot\}_r$ is compatible with $\mathcal{C}_T$.
- A Poisson–Lie bracket on $G$ is compatible with $\mathcal{C}_T$ only if it is a scalar multiple $\{\cdot, \cdot\}_r$ for some $r \in \mathbb{R} T$.
Let \( G \) be a simple complex Lie group. For any Belavin-Drinfeld triple \( T = (\Gamma_1, \Gamma_2, \tau) \) there exists a cluster structure \( C_T \) on \( G \) such that

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Main Conjecture

Let \( G \) be a simple complex Lie group. For any Belavin-Drinfeld triple \( T = (\Gamma_1, \Gamma_2, \tau) \) there exists a cluster structure \( C_T \) on \( G \) such that

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Example I: Standard Case

Trivial Belavin-Drinfeld data: $\Gamma_1 = \Gamma_2 = \emptyset$

$\updownarrow$

Standard Poisson-Lie Structure

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Berenstein-Fomin-Zelevinsky cluster structure on double Bruhat cells
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Berenstein-Fomin-Zelevinsky cluster structure on double Bruhat cells

Initial cluster ($GL_n$ case): collection of all trailing dense minors
Standard cluster structure in $GL_5$: initial quiver
Example II: ”Maximal” Belavin-Drinfeld Data

Cremmer-Gervais Poisson Structure

$G = SL_n$

$\Gamma_1 = \{ \alpha_2, ..., \alpha_{n-1} \}$

$\Gamma_2 = \{ \alpha_1, ..., \alpha_{n-2} \}$

$\gamma(\alpha_i) = \alpha_i - 1$ for $i = 2, ..., n-1$.

**Theorem**

There exists a cluster structure $C_{CG}$ on $SL_n/\Gamma_1/\Gamma_2 = Mat_N$ compatible with the Cremmer–Gervais Poisson–Lie structure and satisfying all conditions of the Main Conjecture.

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There exists a cluster structure \( \mathcal{C}_{CG} \) on \( SL_n/GL_n/\text{Mat}_N \) compatible with the Cremmer–Gervais Poisson–Lie structure and satisfying all conditions of the Main Conjecture.
<table>
<thead>
<tr>
<th></th>
<th>Standard</th>
<th>Cremmer-Gervais</th>
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</thead>
<tbody>
<tr>
<td>( {x_{11}, x_{55}} )</td>
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<td>( x_{15}x_{51} + x_{21}x_{45} + x_{25}x_{41} + x_{21}x_{45} + x_{31}x_{35} )</td>
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For $X, Y \in \text{Mat}_n$, let $X = \begin{bmatrix} X & 0 \end{bmatrix}_0$, $Y = \begin{bmatrix} 0 & Y \end{bmatrix}_{1, n-1}$.

Put $k = \lfloor \frac{n+1}{2} \rfloor$, $N = k(n-1)$ and define a $k(n-1) \times (k+1)(n+1)$ matrix $U_{(X, Y)} = \begin{bmatrix} Y & X & 0 & \cdots & 0 \\ 0 & Y & X & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & Y & X \\ 0 & \cdots & 0 & 0 & Y \end{bmatrix}$.

Define $\theta_i(X) = \det X\begin{bmatrix} n-i+1 \\ n-i+1 \end{bmatrix}$, $i \in \begin{bmatrix} n-1 \end{bmatrix}$; $\phi_p(X, Y) = \det U_{(X, Y)}\begin{bmatrix} k(n+1)-p+1 \\ k(n+1) \end{bmatrix}$, $p \in \begin{bmatrix} N \end{bmatrix}$; $\psi_q(X, Y) = \det U_{(X, Y)}\begin{bmatrix} k(n+1)-q+2 \\ k(n+1)+1 \end{bmatrix}$, $q \in \begin{bmatrix} M \end{bmatrix}$. In the last family, $M = N/4$ if $n$ is even/odd.
For $X, Y \in \text{Mat}_n$, let

$$
\mathcal{X} = [X_{[2,n]} \ 0], \quad \mathcal{Y} = [0 \ Y_{[1,n-1]}].
$$

In the last family, $M = \frac{N}{M} = N - n + 1$ if $n$ is even/odd.
For $X, Y \in \text{Mat}_n$, let

$$X = [X_{2,n} \ 0], \quad Y = [0 \ Y_{1,n-1}] .$$

Put $k = \left\lfloor \frac{n+1}{2} \right\rfloor$, $N = k(n - 1)$ and define a $k(n-1) \times (k + 1)(n + 1)$ matrix

$$U(X, Y) = \begin{bmatrix}
Y & X & 0 & \cdots & 0 \\
0 & Y & X & 0 & \cdots \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & Y & X
\end{bmatrix} .$$
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Define

$$\theta_i(X) = \det X_{[n-i+1,n]}^{[n-i+1,n]}, \quad i \in [n-1];$$

$$\varphi_p(X, Y) = \det U(X, Y)_{[N-p+1,N]}^{[k(n+1)-p+1,k(n+1)]}, \quad p \in [N];$$

$$\psi_q(X, Y) = \det U(X, Y)_{[N-q+1,N]}^{[k(n+1)-q+2,k(n+1)+1]}, \quad q \in [M].$$

In the last family, $M = N/N = N - n + 1$ if $n$ is even/odd.
Figure: Translation invariance properties of $\bar{U}(X, X)$
The functions $\theta_i(X), \phi_p(X, X), \psi_q(X, X)$ form a log-canonical family with respect to the Cremmer–Gervais bracket.
Theorem

The functions $\theta_i(X), \phi_p(X, X), \psi_q(X, X)$ form a log-canonical family with respect to the Cremmer–Gervais bracket.

Intuition behind a construction of the initial cluster as well as the method of the proof come from considering the Poisson-Lie (Drinfeld) double of $SL_N$ associated with the Cremmer-Gervais structure.
Figure: Quiver $Q_{CG}(5)$
Theorem

The cluster structure $\mathcal{C}_{CG}$ is regular.
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The cluster structure $\mathcal{C}_{CG}$ is regular.

The proof relies on Dodgson-type identities applied to submatrices of $U(X, Y)$ while taking into account its shift-invariance properties.
Theorem

\[ O(\text{Mat}_n) \subset \tilde{\mathcal{A}}(C_{CG}) \]
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\[ O(\text{Mat}_n) \subset \bar{A}(\mathcal{CG}) \]

The proof relies on induction on \( n \).
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Strategy

Two distinguished sequences of cluster transformations:

\[ S \text{(number of mutations quadratic in } n) - \text{followed by freezing some of the cluster variables and localization at a single cluster variable} \]

\[ \varphi_{n-1}(X) = \det [X] \]

realizes a map \( \zeta : \text{Mat}_n \{ X : \varphi_{n-1}(X) = 0 \} \rightarrow \text{Mat}_{n-1} \) that "respects" the Cremmer–Gervais cluster structure.

\[ T \text{(number of mutations cubic in } n) - \text{realizes the anti-Poisson involution} \]

\[ X \mapsto W_0 X W_0 \]

\( W_0 \) - the longest permutation.

M. Gekhtman (joint with M. Shapiro and A. Vainshtein) (Notre Dame)
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Two distinguished sequences of cluster transformations:

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Further Results and Work in Progress

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\[ \text{TotPos}_{CG}(n) \subsetneq \text{TotPos}(n). \]
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*The cluster algebra* \( \mathcal{A}_{CG}(3) \) *is a proper subalgebra of the upper cluster algebra* \( \overline{\mathcal{A}}_{CG}(3) \).
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The cluster algebra \( \mathcal{A}_{CG}(3) \) is a proper subalgebra of the upper cluster algebra \( \overline{\mathcal{A}}_{CG}(3) \).

Idea of the proof: show that \( x_{12} \) can not belong to a log-canonical coordinate chart w.r.t. the Cremmer-Gervais Poisson structure.
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Conjecture

The cluster algebra \( \mathcal{A}_{CG}(n) \) is a proper subalgebra of the upper cluster algebra \( \overline{\mathcal{A}}_{CG}(n) \).
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For any Belavin-Drinfeld data, there exists a compatible generalized cluster structure on the corresponding Drinfeld double and the dual Poisson-Lie group.
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For any Belavin-Drinfeld data, there exists a compatible generalized cluster structure on the corresponding Drinfeld double and the dual Poisson-Lie group.

Proved for both the standard and Cremmer-Gervais cases in $GL_n$.

General $GL_n$ case: proof in progress.
Example in $GL_8$:
Initial quiver for the standard double of $GL_4$:


Thank you!