

Math 125A Homework, with answers

Homework assignment (due Thurs May 8th):

There are 2 problems.

Problem (1)

(This is about connectivity in graphs)

For M a set with a binary relation $R \subset M \times M$, we say that M, R is connected *iff_{def}* for all $a, a' \in M$ there is $n = 0, 1, \dots$ and there are b_0, b_1, \dots, b_n such that $a = b_0, a' = b_n$ and for $i = 0, 1, \dots, n-1$, either $\langle b_i, b_{i+1} \rangle \in R$ or $\langle b_{i+1}, b_i \rangle \in R$.

Let P be a 2-ary relation symbol.

(a) Show there are formulas $\varphi_n(x, y)$ in $\mathcal{L}_{\{P\}}$ ($n = 1, 2, \dots$) such that for all $\mathcal{L}_{\{P\}}$ -structures \mathcal{M}

$M, I_{\mathcal{M}}(P)$ is connected *iff* for all $a, a' \in M$, there is n st $\mathcal{M} \models \varphi_n(a, a')$.

(a) Answer: Let $\varphi_n(x, y)$ be
 $\exists z_0 \exists z_1 \dots \exists z_n ((x \hat{=} z_0) \wedge (\bigwedge_{i=0}^{i=n-1} (P(z_i, z_{i+1}) \vee P(z_{i+1}, z_i))) \wedge (z_n = y)).$

(b) Show that for all \mathcal{M} structures,
 either there is $\mathcal{M}_1 \equiv \mathcal{M}$ such that \mathcal{M}_1 is not connected,
 or there is m st $\mathcal{M} \models \forall x \forall y (\varphi_1(x, y) \vee \varphi_2(x, y) \vee \dots \vee \varphi_m(x, y)).$

(b) Answer:

Let $Th(\mathcal{M}) = \{\psi; \psi \text{ is a sentence of } \mathcal{L}_{\{P\}} \text{ and } \mathcal{M} \models \psi\}.$

Let c, d be two constants. Let $U = \{\neg \varphi_n(c, d); n = 1, 2, \dots\}.$

Let $T = Th(\mathcal{M}) \cup U.$

If T is satisfiable then there is $\mathcal{M}_1^* \models T.$ Let $a, a' \in M_1^*$ be the interpretations of $c, d.$ So \mathcal{M}_1^* is a $\mathcal{L}_{\{P\}}$ structure \mathcal{M}_1 which interpretes c as a and d as $a'.$

Since $\mathcal{M}_1^* \models U$ we have $\mathcal{M}_1 \models \neg \varphi_n(a, a')$ (for $n = 1, 2, \dots$), so by (i) \mathcal{M}_1 is not connected.

If T is not satisfiable, there is a finite $T_0 \subseteq T$ st T_0 is not satisfiable. There is a m st $T_0 \subseteq Th(\mathcal{M}) \cup \{\neg \varphi_n(c, d); n = 1, 2, \dots, m\}.$

So, for all a, a' in M , if we let \mathcal{M}^* be the $\mathcal{L}_{\{P, c, d\}}$ structure obtained from \mathcal{M} by interpreting c, d as $a, a',$ we have $\mathcal{M}^* \not\models T_0.$ But $\mathcal{M}^* \models Th(\mathcal{M});$ so for some $n \leq m,$ $\mathcal{M}^* \models \varphi_n(c, d);$ thus $\mathcal{M} \models \varphi_n(a, a').$

This held for all $a, a' \in M;$ so

$\mathcal{M} \models \forall x \forall y (\varphi_1(x, y) \vee \varphi_2(x, y) \vee \dots \vee \varphi_m(x, y)).$

Problem (2) is on the next page:

Problem (2)

(This is about coloring graphs)

For M a set with a binary relation $R \subset M \times M$, we say that M, R is 4-colorable iff_{def} there are sets $B_1, B_2, B_3, B_4 \subseteq M$ such that:

$$B_1 \cup B_2 \cup B_3 \cup B_4 = M, \text{ and}$$

for each $i = 1, 2, 3, 4$: for all $a, b \in B_i$, if $a \neq b$ then $\langle a, b \rangle \notin R$ and $\langle b, a \rangle \notin R$.

(We say that such B_1, B_2, B_3, B_4 are a 4-coloring of M, R)

Let Q_1, Q_2, Q_3, Q_4 be unary relation symbols, and let P be a binary relation. Let $\mathcal{A} = \{P, Q_1, Q_2, Q_3, Q_4\}$.

(a) Write down a finite set *COLOR* of sentences of $\mathcal{L}_{\mathcal{A}}$ st for all $\mathcal{L}_{\mathcal{A}}$ -structures \mathcal{M} ,

letting $R = I_{\mathcal{M}}(P)$, and letting $B_i = I_{\mathcal{M}}(Q_i)$ ($i = 1, 2, 3, 4$).

$\mathcal{M} \models \textit{COLOR}$ iff B_1, B_2, B_3, B_4 is a 4-coloring of M, R .

(a) Answer:

$\forall x (\bigvee_{i=1}^4 (Q_i(x)))$ and (for each $i = 1, \dots, 4$) we have sentence:
 $\forall x \forall y ((Q_i(x) \wedge Q_i(y) \wedge \neg(x \hat{=} y)) \rightarrow \neg(P(x, y)))$

(b) Show that for any N, R and any $M \subseteq N$, if N, R is 4-colorable then M, R' is 4-colorable.

(where $R' =$ the restriction of R to M which equals_{def} $R \cap M \times M$).

(b) Answer: If B_1, \dots, B_4 show that N, R is 4-colorable, then $(B_1 \cap M), \dots, (B_4 \cap M)$ shows M, R' is 4-colorable.

(c) Given a $\mathcal{L}_{\{P\}}$ -structure \mathcal{M} , for each $a \in M$, let c_a be a constant symbol (where $a \neq a'$ implies c_a and $c_{a'}$ are different constant symbols). Let $\mathcal{A}' = \{P\} \cup \{c_a ; a \in M\}$.

Show there is a set $T_{\mathcal{M}}$ of $\mathcal{L}_{\mathcal{A}'}$ sentences such that for all $\mathcal{L}_{\mathcal{A}'}$ -structures \mathcal{N}

$\mathcal{N} \models T_{\mathcal{M}}$ iff the function $a \mapsto I_{\mathcal{N}}(c_a)$ maps \mathcal{M} isomorphically to a substructure of $(N, I_{\mathcal{N}}(P))$.

(c) Answer: Let $T_{\mathcal{M}}^0 = \{\neg(c_a \hat{=} c_b) ; a, b \in M, a \neq b\}$
 Let $T_{\mathcal{M}}^+ = \{P(c_a, c_b) ; a, b \in M, \langle a, b \rangle \in I_{\mathcal{M}}(P)\}$
 Let $T_{\mathcal{M}}^- = \{\neg(P(c_a, c_b)) ; a, b \in M, \langle a, b \rangle \notin I_{\mathcal{M}}(P)\}$

Let $T_{\mathcal{M}} = T_{\mathcal{M}}^0 \cup T_{\mathcal{M}}^+ \cup T_{\mathcal{M}}^-$.

Given a $\mathcal{L}_{\mathcal{A}'}$ structure \mathcal{N} ,

if $\mathcal{N} \models T_{\mathcal{M}}$ then, for the map h , from M to N , given by $h(a) = I_{\mathcal{N}}(c_a)$:

$\mathcal{N} \models T_{\mathcal{M}}^0$ iff h is 1-1;

$\mathcal{N} \models T_{\mathcal{M}}^+$ iff $h(I_{\mathcal{M}}(P)) \subseteq I_{\mathcal{N}}(P)$;

$\mathcal{N} \models T_{\mathcal{M}}^-$ iff $h(M^2 \sim I_{\mathcal{M}}(P)) \subseteq (N^2 \sim I_{\mathcal{N}}(P))$;

So $\mathcal{N} \models T_{\mathcal{M}}$ iff h maps \mathcal{M} isomorphically to a substructure of $(N, I_{\mathcal{N}}(P))$.

(d) Show: if \mathcal{M} is a $\mathcal{L}_{\{P\}}$ -structure and if every finite substructure of \mathcal{M} is 4-colorable then \mathcal{M} is 4-colorable.

(Hint: use compactness to see that $T_{\mathcal{M}} \cup COLOR$ is satisfiable; for $\mathcal{N} \models T_{\mathcal{M}} \cup COLOR$, by (a) \mathcal{N} is 4-colorable; so by (c) and (b) \mathcal{M} is isomorphic with a 4-colorable structure and so \mathcal{M} is 4-colorable).

(d) Answer: Let $\Gamma = T_{\mathcal{M}} \cup COLOR$.

For any finite $\Gamma_0 \subseteq \Gamma$ there is a finite substructure \mathcal{M}_0 of \mathcal{M} st $\Gamma_0 \subseteq COLOR \cup T_{\mathcal{M}_0}$.

Since \mathcal{M}_0 is 4-colorable (say by $B_i^0, i = 1, \dots, 4$), the structure \mathcal{M}_0 with B_i^0 interpreting Q_i will satisfy (by (a) and (c)) $COLOR \cup T_{\mathcal{M}_0}$.

Thus Γ is satisfiable (by compactness). If $\mathcal{N} \models \Gamma$ then by (c) \mathcal{M} is isomorphic to a $\mathcal{L}_{\{P\}}$ substructure \mathcal{M}_1 of \mathcal{N} . Since $\mathcal{N} \models COLOR$, the $\mathcal{L}_{\{P\}}$ structure $N, I_{\mathcal{N}}(P)$ is 4-colorable. So by (b), \mathcal{M}_1 is 4-colorable.

But then \mathcal{M} is 4-colorable: if h is the isomorphism from \mathcal{M} to \mathcal{M}_1 and if $B_i^1, i = 1, \dots, 4$ shows \mathcal{M}_1 is 4-colorable, then (for $B_i = h^{-1}(B_i^1)$), $B_i, i = 1, \dots, 4$ shows \mathcal{M} is 4-colorable.