

Math 125A Midterm

There are 7 problems worth a total of 100 points.

There are 80 minutes available.

Problem 1 (10 points)

For t a term of $\mathcal{L}_{\mathcal{A}}$, let $a(t) = a_1 + \dots + a_m$ where t has m occurrences of function symbols, and the i th occurrence is a function symbol of arity a_i .

For t a term of $\mathcal{L}_{\mathcal{A}}$, let $s(t) =$ the number of occurrences of symbols in t , not counting occurrences of (and).

Show: for all terms t , $s(t) = 1 + a(t)$.

Problem 2 (15 points)

Let $\mathcal{A} = \{G, H\}$ where G is a 1-ary function symbol and H is a 2-ary function symbol. Let \mathcal{M} be the $\mathcal{L}_{\mathcal{A}}$ structure where: $M = \{T, F\}$ and $I_{\mathcal{M}}(G)$ is the truth function (connective) for \neg , and $I_{\mathcal{M}}(H)$ is the truth function (connective) for \rightarrow . Let $\mathcal{T} =$ the set of terms of $\mathcal{L}_{\mathcal{A}}$.

For $f : \{1, 2, \dots\} \rightarrow M$, let v_f be the truth assignment given by: $v_f(A_i) = f(i)$; and let w_f be the assignment (to variables) given by: $w_f(x_i) = f(i)$.

Show that there is a function ρ from \mathcal{T} to \mathcal{L}_0 such that:

for all $f : \{1, 2, \dots\} \rightarrow M$ and all t in \mathcal{T} , $\overline{v_f}(\rho(t)) = \overline{w_f}(t)$.

(you should specify the function ρ by a recursion).

Problem 3 (15 points)

(This continues problem (2)):

For each formula φ of $\mathcal{L}_{\mathcal{A}}$, find a formula φ^* of \mathcal{L}_0 such that, for all $f : \{1, 2, \dots\} \rightarrow M$ $(\mathcal{M}, w_f) \models \varphi$ iff $\overline{v_f}(\varphi^*) = T$.

Problem 4 (15 points)

Let \mathcal{A} be a set of function, relation and constant symbols, where G is a 1-ary function symbol not in \mathcal{A} . Let $\mathcal{A}^* = (\mathcal{A} \cup \{G\})$.

Given a $\mathcal{L}_{\mathcal{A}}$ structure \mathcal{N} and a automorphism e of \mathcal{N} , let \mathcal{N}^* be the $\mathcal{L}_{\mathcal{A}^*}$ structure given by: $N^* = N$; for s in \mathcal{A} $I_{\mathcal{N}^*}(s) = I_{\mathcal{N}}(s)$; and $I_{\mathcal{N}^*}(G) = e$.

Show: e is an automorphism of \mathcal{N}^* .

Problem 5 (15 points)

Let $\mathcal{A} = \{E\}$ where E is a 2-ary relation symbol. A $\mathcal{L}_{\mathcal{A}}$ structure \mathcal{M} is a graph iff_{def} $\mathcal{M} \models \forall x((\neg(E(x, x))) \wedge \forall x \forall y(E(x, y) \rightarrow E(y, x)))$

Consider the following graph \mathcal{M} :

$M = \{1, 2, 3, 4, 5\}$;

$I_{\mathcal{M}}(E)$ is $\{\langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 3, 4 \rangle, \langle 4, 3 \rangle, \langle 4, 5 \rangle, \langle 5, 4 \rangle\}$.

For each of the elements a, b listed below, either exhibit an automorphism e of \mathcal{M} such that $e(a) = b$; or exhibit a formula $\varphi(x)$ of $\mathcal{L}_{\mathcal{A}}$ such that $\mathcal{M} \models \varphi(a)$ and $\mathcal{M} \not\models \varphi(b)$.

(i) $a = 1, b = 2$; (ii) $a = 1, b = 3$; (iii) $a = 3, b = 4$.

Problem 6 (15 points)

Let Γ_1 and Γ_2 be two sets of formulas from \mathcal{L}_0 . Suppose there is no truth assignment v such that $v \models \Gamma_1$ and $v \models \Gamma_2$.

Show that there is a formula φ in \mathcal{L}_0 such that $\Gamma_1 \models \varphi$ and $\Gamma_2 \models (\neg\varphi)$.

Problem 7 (15 points)

Let Γ be a set of formulas from \mathcal{L}_0 . Assume there is a finite set F of formulas from \mathcal{L}_0 such that F is logically equivalent to Γ .

Show that there is a finite *subset* Γ_0 of Γ such that Γ_0 is logically equivalent to Γ .