A Crash Course on Stable Range, Cancellation, Substitution, and Exchange

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Abstract

The themes of cancellation, internal cancellation, substitution, and exchange have led to a lot of interesting research in the theory of modules over commutative and noncommutative rings. This article provides a quick and relatively self-contained introduction to the voluminous work in this area, using the notion of the stable range of rings as a unifying tool. With only a small number of exceptions, all theorems stated here are proved in full, modulo basic facts in the theory of modules and rings available in standard textbooks on ring theory.

Introduction

In the study of any algebraic system in which there is a notion of a direct sum, the theme of cancellation arises very naturally: if $A \oplus B \cong A \oplus C$ in the given system, can we conclude that $B \cong C$? (For an early treatment of this problem, see the work of Jónsson and Tarski [JT] in 1947.) The answer is, perhaps not surprisingly, sometimes "yes" and sometimes "no": it all depends on the algebraic system, and it depends heavily on the choice of A as well.

Starting with a simple example, we all know that, by the Fundamental Theorem of Abelian Groups, the category of finitely generated abelian groups satisfies cancellation. But a little more is true, which solved what would have been the "Third Test Problem" for §6 in Kaplansky's book [Ka_1] (see the Notes in [Ka_1: §20]): if A is a f.g. (finitely generated) abelian group, then for *any* abelian groups B and C, $A \oplus B \cong A \oplus C$ still implies $B \cong C$. Thus, f.g. abelian groups A remain "cancellable" (with respect to direct sums) in the category of all abelian groups. This takes a proof, which was first given, independently, by P. M. Cohn [Co] and E. A. Walker [W]. And yet, there exist many torsionfree abelian groups of rank 1 (that is, nonzero subgroups of the rational numbers $\mathbb{Q}$) that are *not* cancellable in the category of torsionfree abelian groups of finite rank, according to B. Jónsson [Jó].

In this article, we are interested in the cancellation problem in categories of modules over rings. The case of abelian groups (or $\mathbb{Z}$-modules) provides an important prototype.

However, new complications arise when the ground ring is no longer the ring of the integers. For instance, even the case of cancellation within the category $\mathcal{C}$ of f.g. projective modules becomes highly nontrivial, say over commutative affine algebras, or over integral group rings of finite groups. Just to cite an explicit example, given Hilbert's Syzygy Theorem, Serre's famous conjecture (ca. 1955) on the freeness of f.g. projective modules over a polynomial ring $R = k[x_1, \ldots, x_n]$ (for a field k) boiled down to a statement about the cancellability of R (the free module of rank 1) in the category $\mathcal{C}$, and it took some 20 years before an affirmation of this was given by Quillen and Suslin (see [La$_6$]).

In general, given any (skeletally small) category of modules $\mathcal{C}$ that is closed with respect to direct sums, one can form the Grothendieck group $K_0(\mathcal{C})$ of the semigroup of isomorphism classes of the modules in $\mathcal{C}$ (where $[A]+[B]$ is defined to be $[A \oplus B]$). If $A \oplus B \cong A \oplus C$ in $\mathcal{C}$, then $[A]+[B] = [A]+[C]$ implies that $[B] = [C] \in K_0(\mathcal{C})$, since $K_0(\mathcal{C})$ is a group. Thus, B and C become the same "upon stabilization" to $K_0(\mathcal{C})$, but this says nothing about the existence or nonexistence of an isomorphism between B and C. From this standpoint, we can think of the cancellation problem $A \oplus B \cong A \oplus C \overset{?}{\Longrightarrow} B \cong C$ as a problem in "nonstable K-theory".

There are several variations on the notion of cancellation. For instance, for a given module A, if $A = K \oplus N = K' \oplus N'$ with $N \cong N'$, does it follow that $K \cong K'$? If the answer is always "yes", A is said to satisfy *internal cancellation* (or A is internally cancellable). Another variation is the following: if B and C are direct summands of a module M with complementary summands isomorphic to a given module A, do B, C necessarily have a *common* direct complement in M? If the answer is always "yes", the given module A is said to have the *substitution property.* Somewhat more naively, we can also ask if a module A can never be isomorphic to a proper direct summand of itself; if this is the case, A is said to be a *Dedekind-finite* module. These properties are easily seen to be related as follows (see §4):

$(*)$ Substitution $\Longrightarrow$ Cancellation $\Longrightarrow$ Internal Cancellation $\Longrightarrow$ Dedekind-Finite.

The hierarchy of the four properties above provides the framework for this article, in which we give an exposition on some of the main research done around these themes. A unifying tool for our presentation is the notion of the stable range of rings, which we introduce first and foremost in §1. As it turns out, what we really need is the basic notion of rings with stable range 1. In §2, we highlight the notion of an internally cancellable module by showing that a von Neumann regular ring R (as a module over itself) satisfies internal cancellation iff R is unit-regular, and this is also shown to be equivalent to R having stable range 1 ((2.9) and (5.5)). Examples of cancellation and non-cancellation are given in §3, before we take up the substitution property in §4, proving that a module has such a property iff its endomorphism ring has stable range 1. Various concrete results on cancellation important for applications are collected in §5 after the statement of Evans' Cancellation Theorem (5.1), and a proof for the remarkable Morita invariance of stable range 1 is given in (5.6).

In 1964, in studying the refinements of direct decompositions of algebraic systems, Crawley and Jónsson introduced the notion of "exchange" and "finite exchange" in their classic paper [CJ] (cf. also [JT]). Some ten years later, in the hands of Warfield, Nicholson, Harada, and others, the exchange method specialized to modules blossomed into a fruitful direction of research in ring theory, and led to the introduction of the important class of *exchange rings.* In §6, we give an exposition of this theory, and present its basic connections to cancellation, substitution, and other earlier themes in this paper. This is followed by §7, in which the theory of exchange modules is further specialized and applied to the case of quasi-injective modules. As a sort of summary for the paper, we offer in §8 a somewhat philosophical discussion on the nature of module-theoretic properties, calling special attention to those (dubbed "ER-properties") that depend only on the endomorphism rings of the modules in question. The paper concludes with an Epilogue (§9), which provides a guide to the reader for further reading on some of the topics discussed here.

While most results in §§1-7 have appeared in original papers before, they are scattered in the literature, and have not been collected together in one place as a coherent body of mathematical knowledge. This is the raison d' être for the present article, which is based on two tutorial lectures given by the author at the Ring Theory Center of Ohio University, Athens, Ohio, on May 21-22, 2003. The initial version of the article, circulated as lecture notes for the tutorial course, consisted of some 20 pages, which are fleshed out here into a more comprehensive exposition on the basic themes contained in the title of this article. I am grateful to my colleagues at the Ring Theory Center, especially Professors Dinh Van Huynh, S. K. Jain, and Sergio López-Permouth, for inviting me to give a tutorial course there, without which this article would clearly never have been written.

§1. Stable Range of Rings

The concept of stable range was initiated by H. Bass in his investigation of the stability properties of the general linear group in algebraic K-theory [Ba]. In ring theory, stable range provides an arithmetic invariant for rings that is related to interesting issues such as cancellation, substitution, and exchange. The simplest case of stable range 1 has especially proved to be important in the study of many ring-theoretic topics. Let us start with the basic notion of the reduction of a unimodular sequence.

Definition 1.1. A sequence $\{a_1, \dots, a_n\}$ in a ring R is said to be *left unimodular* if $Ra_1 + \cdots + Ra_n = R$. In case $n \geq 2$, such a sequence is said to be *reducible* if there exist $r_1, \dots, r_{n-1} \in R$ such that $R(a_1 + r_1 a_n) + \cdots + R(a_{n-1} + r_{n-1} a_n) = R$.

This reduction notion leads directly to the definition of stable range.

Definition 1.2. A ring R is said to have *left stable range* $\leq n$ if every left unimodular sequence of length $> n$ is reducible. The smallest such n is said to be the left stable range of R; we write simply $\mathrm{sr}_\ell(R) = n$. (If no such n exists, we say $\mathrm{sr}_\ell(R) = \infty$.) The right stable range is defined similarly, and is denoted by $\mathrm{sr}_r(R)$.

Proposition 1.3. *To check that* $\mathrm{sr}_\ell(R) \le n$*, it suffices to show that every left unimodular sequence* $\{a_1, \ldots, a_{n+1}\}$ *is reducible.*

Proof. Suppose the last condition holds, and consider a left unimodular sequence $\{a_1, \ldots, a_{n+k}\}$ $(k \ge 1)$. Fix an equation $x_1a_1 + \cdots + x_{n+k}a_{n+k} = 1$. Then

$$\{a_1, \ldots,\ a_n,\ x_{n+1}a_{n+1} + \cdots + x_{n+k}a_{n+k}\}$$

is left unimodular, so we can reduce it to, say:

$$\{\, a_1 + y_1(x_{n+1}a_{n+1} + \cdots + x_{n+k}a_{n+k}),\ \ldots\ ,\ a_n + y_n(x_{n+1}a_{n+1} + \cdots + x_{n+k}a_{n+k})\,\}.$$

Now tag on $a_{n+1}, \ldots, a_{n+k-1}$ to get a left unimodular sequence of length $n+k-1$, and subtract left multiples of these from the earlier entries to get a left unimodular sequence

$$\{\, a_1 + y_1x_{n+k}a_{n+k},\ \ldots\ ,\ a_n + y_nx_{n+k}a_{n+k},\ \ a_{n+1},\ \ldots\ ,\ a_{n+k-1}\,\},$$

which gives the desired reduction for $(a_1, \ldots, a_{n+k})$. □

Remark 1.4. Vaserstein has proved that $\mathrm{sr}_\ell(R) = \mathrm{sr}_r(R)$ for any ring R [Va$_1$]. Thus, we may write $\mathrm{sr}(R)$ for this common value, and call it simply the stable range of R. We'll need Vaserstein's result only in the case of stable range 1: this special case will be dealt with below in (1.8).

The easy proof of the next proposition will be left to the reader.

Proposition 1.5. (1) *If* S *is a factor ring of* R*, then* $\mathrm{sr}(S) \le \mathrm{sr}(R)$.
(2) $\mathrm{sr}(R) = \mathrm{sr}(R/\mathrm{rad}(R))$, where $\mathrm{rad}(R)$ denotes the Jacobson radical of R.

Examples 1.6 (without proofs).

(1) (Bass [Ba]) If R is a commutative noetherian ring of dimension d, then $\mathrm{sr}(R) \le d+1$.
(2) $\mathrm{sr}(\mathbb{Z}) = 2$. (We have $\mathrm{sr}(\mathbb{Z}) \le 2$ by (1), and the irreducibility of $\{2,\ 5\}$ shows that $\mathrm{sr}(\mathbb{Z}) \ne 1$.)
(3) (Vaserstein [Va$_1$]) For any field $k \subseteq \mathbb{R}$, $\mathrm{sr}\big(k[x_1, \ldots, x_n]\big) = n+1$.
(4) For any field k, $\mathrm{sr}\big(k[[x_1, \ldots, x_n]]\big) = 1$. (This follows easily from (1.5)(2).)

We now specialize to the case of stable range 1. The following is a basic observation of Kaplansky [Ka$_2$].

Lemma 1.7. *Suppose* $\mathrm{sr}_\ell(R) = 1$*. Then* R *is Dedekind-finite,*[1] *and* $Ra + Rb = 1 \Longrightarrow a + rb \in \mathrm{U}(R)$ *for some* $r \in R$.

Proof. Suppose $ac = 1$. Then $Ra+R(1-ca) = R$ implies that some $u := a+s(1-ca)$ is left-invertible. Right-multiplying by c, we get $uc = ac+s(c-cac) = 1$. Thus, $u \in \mathrm{U}(R)$, and hence $c \in \mathrm{U}(R)$. The last conclusion is now clear. □

[1]A ring R is called Dedekind-finite if $ac = 1 \in R$ implies that $c \in \mathrm{U}(R)$ (the unit group of R).

Theorem 1.8. ([Va$_1$]) *If* $\mathrm{sr}_r(R) = 1$, *then* $\mathrm{sr}_\ell(R) = 1$ *(and, of course, conversely).*

Proof. (Following [Go$_2$]; see also [La$_2$: Ex. 1.25].) Start with $Rb + Rd = R$. Then $ab + c = 1$ for some $c \in Rd$. From $aR + cR = R$, we have a right invertible element $u := a + cx$ (for some $x \in R$). Say $uv = 1$. For $w := a + x(1 - ba)$, we have

$$\begin{aligned} w(1 - bx) &= a + x(1 - ba) - abx - xb(1 - ab)x \\ &= a + x - xba - abx - xb(u - a) \\ &= a + x - abx - xbu = a + cx - xbu \\ &= (1 - xb)u. \end{aligned}$$

Therefore, for $y := (1 - bx)v$, we have $wy = 1 - xb$. It follows that

$$\begin{aligned} w(b + yc) &= ab + x(1 - ba)b + (1 - xb)c \\ &= ab + xbc + (1 - xb)c = 1. \end{aligned}$$

Thus, $R(b + yc) = R$, with $yc \in yRd \subseteq Rd$, as desired. ☐

The above proof is based on *ad hoc* algebraic calculations. A more conceptual proof of (1.8) is also possible; see the proof of (4.4) below.

In (5.4)(2) below, we shall prove that if a ring R has stable range 1, then so does any matrix ring $\mathbb{M}_n(R)$. Assuming this result in advance, we can give several other interesting characterizations for rings of stable range 1, due to M. Canfell [Ca: Th. 2.9]. (Here, we work with right ideals and right modules in order to avoid using the transpose of matrices in the proof.)

Theorem 1.9. *For any ring* R, *the following are equivalent:*

(1) $\mathrm{sr}(R) = 1$.
(2) *If* $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_n$ *are two finite sets of generators for a right* R*-module* M, *there exists a matrix* $U \in \mathrm{GL}_n(R)$ *such that* $(a_1, \ldots, a_n) = (b_1, \ldots, b_n) \cdot U$.
(3) *Condition* (2) *for* $n = 1$.
(4) *For* $a, b, c \in R$, $aR + bR = cR \Longrightarrow a + br = cu$ *for some* $r \in R$ *and* $u \in \mathrm{U}(R)$.
(5) *For* $a, b \in R$, *if* $aR + bR$ *is principal, then* $aR + bR = (a + br)R$ *for some* $r \in R$.

Proof. (1) $\Longrightarrow$ (2) (following [Ch$_1$]). Let A, $B \in \mathbb{M}_n(R)$ be such that $(a_1, \ldots, a_n) \cdot A = (b_1. \ldots, b_n)$, and $(b_1, \ldots, b_n) \cdot B = (a_1. \ldots, a_n)$. In $\mathbb{M}_n(R)$, we have $BA + (I_n - BA) = I_n$, so the pair $\{B,\, I_n - BA\}$ is right unimodular. Since $\mathrm{sr}_r(\mathbb{M}_n(R)) = 1$ (by (5.4)(2) below), we have $U := B + (I_n - BA)\,C \in \mathrm{GL}_n(R)$ for some $C \in \mathbb{M}_n(R)$. From $(b_1, \ldots, b_n) \cdot BA = (b_1, \ldots, b_n)$, we get

$$(b_1, \ldots, b_n) \cdot U = (b_1, \ldots, b_n) \cdot [\, B + (I_n - BA)\,C\,] = (a_1, \ldots, a_n).$$

(2) $\Longrightarrow$ (3) is a tautology.
(3) $\Longrightarrow$ (4). Given $aR + bR = cR$, consider the factor R-module $M = cR/bR$ and let "bar" denote the quotient map from cR onto M. Since $M = \overline{c}\,R = \overline{a}\,R$, (3) implies

that $\overline{a} = \overline{c}u$ for some $u \in \mathrm{U}(R)$. Therefore, $cu - a = br$ for some $r \in R$, which proves(4).

(4) $\Longrightarrow$ (5) is clear, since in (4), $u \in \mathrm{U}(R)$ implies that $cR = cuR = (a+br)R$.

(5) $\Longrightarrow$ (1) is clear by applying (5) in the special case $aR + bR = R$. $\square$

§2. Unit-Regular Rings and Internal Cancellation

An element $a \in R$ is said to be (von Neumann) regular iff $a = axa$ for some $x \in R$. If all elements of R are regular, we say R is (von Neumann) regular. Such rings can also be characterized by the fact that principal 1-sided ideals split in R: see [La$_1$: (4.23)].

The following criterion for a module endomorphism to be regular is well-known and easy to verify; see, e.g. the solution to [La$_2$: Ex. 4.14A$_1$].

Proposition 2.1. *Let $R = \mathrm{End}(M_k)$, where M_k is a right module over the ring k. Then $a \in R$ is regular iff $\ker(f)$ and $\mathrm{im}(f)$ are both direct summands of M_k.*

Corollary 2.2. *The endomorphism ring of any semisimple module M_k is regular.*

Following G. Ehrlich, we say that a ring R is *unit-regular* if every element $a \in R$ is unit-regular; that is, a can be written as aua for some $u \in \mathrm{U}(R)$. Such rings are, in particular, von Neumann regular.

Ehrlich-Handelman Theorem 2.3. ([Eh], [Ha]) *Let M_k be a module such that $R = \mathrm{End}(M_k)$ is regular. Then R is unit-regular iff, whenever $M = K \oplus N = K' \oplus N'$ (in the category of k-modules), $N \cong N' \Rightarrow K \cong K'$.*

Proof. For the "if" part, consider $a \in R$. By (2.1), we have

$$M = \ker(a) \oplus P = Q \oplus \mathrm{im}(a)$$

for suitable k-submodules $P, Q \subseteq M$. Since a defines an isomorphism from P to $\mathrm{im}(a)$, the hypothesis implies that $\ker(a) \cong Q$. Defining $u \in \mathrm{U}(R)$ such that u is an isomorphism from Q to $\ker(a)$, and $u : \mathrm{im}(a) \to P$ is the inverse of $a|P :\ P \to \mathrm{im}(a)$, we have $a = aua \in R$.

For the "only if" part, assume R is unit-regular. Suppose $M = K \oplus N = K' \oplus N'$, where $N \cong N'$. Define $a \in R$ such that $a(K) = 0$ and $a|N$ is a fixed isomorphism from N to N'. Write $a = aua$, where $u \in \mathrm{U}(R)$. We see easily that

$$M = \ker(a) \oplus \mathrm{im}(ua) = K \oplus u(N'). \tag{2.4}$$

Since u defines an isomorphism from N' to $u(N')$, it induces an isomorphism from M/N' to $M/u(N')$. Noting that $M/N' \cong K'$ and $M/u(N') \cong K$ (from (2.4)), we conclude that $K \cong K'$. $\square$

The condition on M_k in the theorem above can be turned into a definition.

Definition 2.5. We say that a module M_k satisfies *internal cancellation* (or M is *internally cancellable*) if, whenever $M = K \oplus N = K' \oplus N'$ (in the category of k-modules), $N \cong N' \Rightarrow K \cong K'$.

An obvious necessary condition for M_k to satisfy internal cancellation is that it is *Dedekind-finite*, in the sense that $M \cong M \oplus X \Rightarrow X = 0$.[2] In general, however, this condition is only necessary, but not sufficient.

Corollary 2.6. (1) *A regular ring R is unit-regular iff R_R satisfies internal cancellation.* (2) *Any semisimple ring is unit-regular.*

Proof. (1) Apply (2.3) to $M = R_R$, and note that $\mathrm{End}(R_R) \cong R$. (2) follows quickly from (1) since a semisimple ring R is regular, and R_R satisfies internal cancellation by an easy application of the Jordan-Hölder Theorem. $\square$

Recall that a ring R is *abelian* if all idempotents in R are central.

Theorem 2.7. *Abelian regular rings are unit-regular. (In particular, commutative regular rings are unit-regular.)*

Proof. Say R is abelian regular. Suppose $R = K \oplus N = K' \oplus N'$, where K, N, K', N' are right ideals, with $N \cong N'$. As is well-known, we have

$$N = eR, \quad K = (1-e)R, \quad \text{and} \quad N' = e'R, \quad K' = (1-e')R,$$

for suitable idempotents $e,\ e' \in R$. By assumption, $e,\ e'$ are *central.* Taking annihilators of the right R-modules $N \cong N'$, we get $(1-e)R = (1-e')R$, so in fact $K = K'$. By (2.6)(1), R is unit-regular. (More precisely, one can actually show that, in the above situation, $e = e'$: see [La$_2$: Ex. 22.2].) $\square$

Remark 2.8. It is known that abelian regular rings are precisely the reduced regular rings, or rings R in which $a \in a^2R$ for every $a \in R$ (a.k.a. strongly regular rings); see [Go$_3$: Ch. 3]. According to (2.7), all such rings are unit-regular.

We now come to the following result proved independently (and at around the same time) by Fuchs [Fu$_2$], Kaplansky [Ka$_2$], and Henriksen [He].

Theorem 2.9. *If R is a unit-regular ring, then* $\mathrm{sr}(R) = 1$. (*In particular, such a ring must be Dedekind-finite, by* (1.7).)

Proof. Say $Ra + Rb = R$. The Chinese Remainder Theorem gives an exact sequence

$$0 \longrightarrow Ra \cap Rb \longrightarrow R \longrightarrow \frac{R}{Ra} \oplus \frac{R}{Rb} \longrightarrow 0,$$

[2] This definition is easily seen to be an extension of the definition of Dedekind-finite rings given in Footnote 1. For a more precise statement, see the first part of the proof of (8.5) below.

which splits since the last module is projective. This shows that $Ra \cap Rb$ is *principal*, so it is a direct summand in ${}_RR$. We can thus write $Rb = (Ra \cap Rb) \oplus K$ for a suitable left ideal K, so that $R = Ra \oplus K$. On the other hand, the map f defined by right multiplication by a gives another exact sequence

$$0 \longrightarrow K' \longrightarrow R \xrightarrow{f} Ra \longrightarrow 0, \quad \text{where} \quad K' = \ker(f).$$

Since Ra is projective, this sequence also splits, so we have $Ra \oplus K' \cong R = Ra \oplus K$. By (2.6)(1) (or more precisely, its left analogue), there exists an isomorphism $\theta : K' \to K$. Letting $g : R \to K'$ be a splitting for the inclusion map $K' \hookrightarrow R$, we can compose the isomorphisms

$$R \xrightarrow{(f,g)} Ra \oplus K' \xrightarrow{(1,\theta)} Ra \oplus K = R,$$

to get an automorphism φ of ${}_RR$, taking $r \in R$ to $ra + \theta(g(r))$. Setting $r = 1$, we see that $a + \theta(g(1)) = \varphi(1) \in \mathrm{U}(R)$. This gives what we want, since $\theta(g(1)) \in K \subseteq Rb$.

A more refined version of (2.9) will be proved in (5.5) below. Note that (2.9) essentially subsumes the following theorem of Bass [Ba], which was one of the earliest results obtained on the stable range of rings.

Corollary 2.10. *If R is a semilocal ring (that is, $R/\mathrm{rad}(R)$ is semisimple), then* $\mathrm{sr}(R) = 1$. *(In particular, this conclusion applies to all left or right artinian rings.)*

Proof. This follows from (1.5)(2), (2.6)(2), and (2.9). □

Corollary 2.11. *Let M_k be an infinite-dimensional vector space over a division ring k. Then the endomorphism ring $R = \mathrm{End}(M_k)$ is regular but not unit-regular.*

Proof. The first part follows from (2.1), and the second from (2.9) together with the (easily verified) fact that the ring R is not Dedekind-finite. □

We conclude this section with another result on stable range 1. Note that this result is not covered by (1.6)(1) since the ring in question is *not* assumed to be noetherian.

Proposition 2.12. *Any 0-dimensional commutative ring R has stable range* 1.

Proof. Since $\mathrm{rad}(R)$ is the intersection of all maximal (= prime) ideals, it coincides with $\mathrm{Nil}(R)$. By (1.5)(2), we may thus assume that R is reduced. At any prime ideal $\mathfrak{p}$, the localization $R_{\mathfrak{p}}$ is also reduced, so it is a field. For any $a \in R$, it follows that $aR_{\mathfrak{p}} = a^2 R_{\mathfrak{p}}$, and so $aR = a^2 R$. This shows that R is regular, and hence $\mathrm{sr}(R) = 1$ by (2.7) and (2.9). □

§3. Examples and Properties of Cancellable Modules

For modules A, B, C over a ring k, $A \oplus B \cong A \oplus C$ in general does not imply $B \cong C$. In fact, given non-isomorphic modules B and C, if we let $A := C \oplus B \oplus C \oplus B \oplus \cdots$, then

$B \oplus A \cong C \oplus A$ (since both are $\cong A!$), and we cannot "cancel" A. (This construction is often referred to as "Eilenberg's trick".)

Definition 3.1. An object A in a category of k-modules $\mathcal{A}$ is said to be *cancellable in* $\mathcal{A}$ if, for any objects B, C in $\mathcal{A}$, $A \oplus B \cong A \oplus C$ implies $B \cong C$. (If the category $\mathcal{A}$ is not specified, "cancellable" will be taken to mean cancellable in the category of all k-modules.)

Eilenberg's trick suggests that it is perhaps more fruitful to study the cancellation of f.g. (finitely generated) modules A. But even with f.g. assumptions on all of A, B, C, cancellation is far from automatic. We'll mention some examples below.

Examples 3.2. (1) If the module A fails to be Dedekind-finite, then A is clearly not cancellable.

(2) If a ring is regular but not unit-regular (e.g. the ring in (2.11)), (2.6)(1) implies that there exist cyclic projective modules A, B, C such that $A \oplus B \cong A \oplus C$, but $B \not\cong C$.

(3) Let R be the coordinate ring of the real 2-sphere, i.e. $R = \mathbb{R}[x, y, z]$, with the relation $x^2 + y^2 + z^2 = 1$. Let P be the kernel of the epimorphism $R^3 \to R$ defined by $e_1 \mapsto x$, $e_2 \mapsto y$, and $e_3 \mapsto z$. Then $P \oplus R \cong R^3 \cong R^2 \oplus R$, but a quick argument using the 2-sphere topology shows that $P \not\cong R^2$ (see [La$_6$: (I.4)]). Thus, R is not cancellable in the category of f.g. projective R-modules.

(4) The category of f.g. projective modules over a ring may satisfy cancellation even if R-projectives are not all free. This is the case, for instance, over Dedekind rings, by a simple application of the Steinitz-Chevalley theory of f.g. torsion-free modules over such rings [CR: §22]. (In fact, an even more general fact will be proved below: see (5.8).)

(5) In contrast to (4), it is well-known that *non-f.g.* torsionfree abelian groups need not be cancellable among $\mathbb{Z}$-modules (or even among torsionfree abelian groups of finite rank). This was first shown by the following remarkable example of Bjarni Jónsson [Jó: §2]. Let V be a 3-dimensional $\mathbb{Q}$-vector space with basis $\{x, y, z\}$, and let

$$(*) \qquad x' = 8x + 3y \quad \text{and} \quad y' = 5x + 2y \quad \text{in } V.$$

Then $\{x', y', z\}$ is also a $\mathbb{Q}$-basis for V, with

$$(**) \qquad x = 2x' - 3y' \quad \text{and} \quad y = -5x' + 8y'.$$

Let P_1, P_2 be two *infinite* disjoint sets of prime numbers not containing 5, and let N_i be the set of all squarefree products of the primes in P_i. Now form the following four groups (where $\langle S \rangle$ denotes the subgroup of V generated by a set $S \subseteq V$):

$$A = \langle x/n : \ n \in N_1 \rangle, \qquad A' = \langle x'/n : \ n \in N_1 \rangle,$$
$$B = \langle y/n, \ z/m, \ (y+z)/5 : \ n \in N_1, \ m \in N_2 \rangle, \quad \text{and}$$
$$C = \langle y'/n, \ z/m, \ (3y'+z)/5 : \ n \in N_1, \ m \in N_2 \rangle.$$

Clearly, $A+B$ and $A'+C$ are both *direct* sums, and we can check easily that $A \oplus B = A' \oplus C$, using $(*)$, $(**)$, together with the relations

$$\begin{aligned}(y+z)/5 &= -x' + y' + (3y'+z)/5 \in A' \oplus C,\\ (3y'+z)/5 &= 3x + y + (y+z)/5 \in A \oplus B.\end{aligned}$$

Since obviously $A \cong A'$, we are done if we can show that $B \not\cong C$. To this end, the key observation is that $\pm y$ (resp. $\pm y'$) are special elements in B (resp. C) that can be characterized in the following manner:

$$(\dagger) \qquad \begin{aligned}\{\pm y\} &= \{b \in B : \ \forall n \in \mathbb{N},\ b/n \in B \Leftrightarrow n \in N_1\},\\ \{\pm y'\} &= \{c \in C : \ \forall n \in \mathbb{N},\ c/n \in C \Leftrightarrow n \in N_1\}.\end{aligned}$$

(The verification of these equations is left to the reader.[3]) If there was an isomorphism $\varphi : B \to C$, $(\dagger)$ would imply that $\varphi(y) = \pm y'$, and similarly $\varphi(z) = \pm z$ (with independent signs), leading to

$$\varphi((y+z)/5) = (\varphi(y) + \varphi(z))/5 = \pm(y' \pm z)/5.$$

This is impossible, since a quick analysis of the elements of C shows that $(y' \pm z)/5 \notin C$. Therefore, $B \not\cong C$, as desired.

(6) An R-module P is said to be *stably free* if $P \oplus R^m$ is free for some integer $m \geq 0$. If a ring R admits a f.g. stably free left module P that is not free, then the module ${}_RR$ is easily seen to be non-cancellable. Such an example was given in (3) above. Another example is given by the polynomial ring $R = k[x, y]$ where k is a division ring with two noncommuting elements a, b. In such a ring R, the left ideal P generated by $x + a$ and $y + b$ satisfies $P \oplus R \cong R^2$ but is nonfree, as is shown in [La_6: (II.3)]. Thus, ${}_RR$ is not cancellable.

(7) More generally, Stafford [St] has given a method for constructing nonfree stably free 1-sided ideals for a large class of noncommutative noetherian domains. If R is a domain with two nonzero elements x, y such that $Rx + Ry = R$, then the left ideal $P := Rx \cap Ry$ fits into an exact sequence of R-modules

$$0 \longrightarrow P \xrightarrow{\ \varepsilon\ } Rx \oplus Ry \xrightarrow{\ \pi\ } R \longrightarrow 0,$$

where $\varepsilon(r) = (r,\, r)$, and $\pi(r,\, r') = r - r'$. This (split) exact sequence leads to $P \oplus R \cong Rx \oplus Ry \cong R^2$. If x, y can be chosen such that P is nonfree, then it follows that ${}_RR$ is not cancellable. For a concrete construction, consider the Weyl algebra $R := k\langle x,\, y\rangle$ (with the relation $xy - yx = 1$) over a field k. Since clearly $Rx + Ry = R$, the left ideal $P = Rx \cap Ry$ satisfies $P \oplus R \cong R^2$ as above. In fact, taking the splitting $1 \mapsto (-yx,\, -xy)$ for π, we get a splitting for ε given by

$$(x, 0) \mapsto x^2 y, \quad \text{and} \quad (0, y) \mapsto -y^2 x,$$

[3] We should, however, take this opportunity to point out that it is in this verification that the infinitude of the two prime sets P_1 and P_2 becomes essential!

so that $P = Rx^2y + Ry^2x$. Our job is to show that P is *nonfree*; for this, it suffices to show that P is noncyclic.[4] Since $x^2y = x(yx+1) = (xy+1)x$, we have $P = [R(xy+1) + Ry^2] \cdot x$, so it suffices to show that $Q := R(xy+1) + Ry^2$ is noncyclic. Assume, on the contrary, that $Q = R \cdot f$, where $f \in R$. We think of f as a polynomial of the form $\sum_i f_i(y)x^i$, so we can define $\deg_x(f)$ to be $\max\{i : f_i \neq 0\}$. Write $y^2 = gf$, $xy+1 = hf$, where $g, h \in R$. Since $\deg_x(y^2) = 0$ and f cannot be a constant, we must have $f = ay$ or $f = ay^2$ for some $a \in k$. But then $xy + 1 = hf \in Ry$, and hence $1 \in Ry$, a contradiction.

The existence of nonfree stably free left ideals for Weyl algebras in characteristic 0 was first pointed out by Webber [We]. In fact, Webber proved that *all* left ideals in the Weyl algebra R above are stably free, while Rinehart [R] had noted earlier that R is not a principal left ideal domain. In a similar vein, Stafford [St] showed that enveloping algebras of nonabelian finite-dimensional Lie algebras always possess nonfree stably free left ideals. We should point out, however, that the kind of examples discussed in (6) and (7) above are special to noncommutative rings. If we have $P \oplus R^n \cong R^{n+1}$ over a *commutative* ring R, a quick argument involving exterior powers shows that $P \cong R$; see [La$_6$: (I.4.11)].

Having working through the examples in (3.2) above, let us now say a few words about the relationship between cancellation and internal cancellation. Clearly, the latter does not imply the former: for instance, *any indecomposable module is clearly internally cancellable*, but it may not be cancellable (as the example ${}_RR$ in (3.2)(3) shows). On the other hand, it turns out that "cancellable" does imply "internally cancellable": we'll deduce this from the observation below.

Proposition 3.3. *A module $A \oplus D$ is cancellable iff A and D themselves are.*

Proof. First assume A and D are cancellable. If $(A \oplus D) \oplus B \cong (A \oplus D) \oplus C$, then we can cancel A first and then cancel D, to get $B \cong C$. Conversely, if $A \oplus D$ is cancellable, then from $D \oplus B \cong D \oplus C$, we can add A and cancel $A \oplus D$, to get $B \cong C$. This shows D is cancellable, and by symmetry the same holds for A. □

Proposition 3.4. *If a module A is cancellable, then A is internally cancellable.*

Proof. Say $A = N \oplus K = N' \oplus K'$, with $N \cong N'$. Since N is a direct summand of A, N is also cancellable by (3.3). Thus, from $N \oplus K = N' \oplus K' \cong N \oplus K'$, we get $K \cong K'$. □

Remark 3.5. The proposition (3.3) for cancellable modules begs the question: *what about internally cancellable modules*? The answer is, curiously enough: "yes-and-no"! If a direct sum $A \oplus D$ has internal cancellation, then A and D also do (by a proof similar to the second part of the proof of (3.3)). However, the converse is false. For instance, in

[4] For this reduction, we may use the fact that the ring R is noetherian; see [La$_3$: (1.35)]. However, since our goal is to show that ${}_RR$ is non-cancellable (from the isomorphism $P \oplus R \cong R \oplus R$), showing P to be noncyclic will be sufficient in any case.

the Example (3.2)(3), it is easy to see that R and R^2 both have internal cancellation; but their direct sum R^3 does not.

A very interesting example of cancellation is given by the following result of C. S. Hsü. (This result is a precursor to a more general result in (5.8) below.)

Proposition 3.6. *Let k be a Dedekind domain. Then the module k_k is cancellable.*

Proof. Consider a k-module $M = A \oplus B = A' \oplus C$, where $A \cong A' \cong k$. Projecting M onto A, we get an exact sequence $0 \longrightarrow B \cap C \longrightarrow C \longrightarrow I \longrightarrow 0$, where $I \subseteq A$. Since I is projective, we have $C = (B \cap C) \oplus I'$, where $I' \cong I$. Similarly, $B = (B \cap C) \oplus J'$, where $J' \cong J \subseteq k$. Now $A \oplus J' \cong A' \oplus I'$ (since they are both complements of $B \cap C$ in M). Taking their second exterior powers (or applying (3.2)(4)) yields $J' \cong I'$, and hence $B \cong C$. □

Remark 3.7. (A) The Proposition implies, in particular, that $\mathbb{Z}$ *is cancellable in the category of abelian groups.* However, it is known that $\mathbb{Z}$ is *not* cancellable in the category of groups! This was shown by an example of William Scott, which was included in [W]. Another example was given independently by Hirshon [Hi_1]. On the positive side, Hirshon [Hi_2] has shown that groups whose normal subgroups satisfy the minimal condition are cancellable in the category of groups; in particular, *finite* groups are always cancellable in that category.

(B) Kaplansky has pointed out that, by a slight variation of the proof of (3.6) above, Hsü's theorem can be shown to hold over any commutative hereditary ring k; see [Ka_1: p. 75].[5] On the other hand, P. Ara observed out that (3.6) also holds over a right free ideal ring (a.k.a. right FIR). In such a ring k, the right ideals are free with uniquely determined ranks. Thus, in the proof of (3.6), after reaching the step $A \oplus J' \cong A' \oplus I'$ (where $A \cong A' \cong k$), we can conclude that $J' \cong I'$ by a rank consideration, to complete the proof that the module k_k is also cancellable in this case.

Following Fuchs [Fu_3], we shall derive an interesting property (3.9) of a cancellable module. This is preceded by a preliminary pushout lemma.

Lemma 3.8. *Let $\alpha : S \to X$ and $\beta : S \to Y$ be monomorphisms of k-modules, and let P be their "pushout"; that is, $P = (X \oplus Y)/Z$, where $Z = \{(\alpha(s), -\beta(s)) : s \in S\}$. If there exists a $\gamma \in \mathrm{Hom}_k(X, Y)$ such that $\beta = \gamma\alpha$, then $P \cong Y \oplus (X/\alpha S)$.*

Proof. Let $X' = \{(x, -\gamma(x)) : x \in X\} \subseteq X \oplus Y$. Then $X' \cong X$, and $X \oplus Y = X' \oplus Y$. Clearly, $Z \subseteq X'$ (since $-\beta(s) = -\gamma(\alpha(s))$ for any $s \in S$). Therefore,

$$P = \frac{X \oplus Y}{Z} = \frac{X' \oplus Y}{Z} \cong \frac{X'}{Z} \oplus Y \cong \frac{X}{\alpha S} \oplus Y,$$

where the isomorphism $X'/Z \cong X/\alpha S$ is induced by the natural isomorphism $X \to X'$ given by $x \mapsto (x, -\gamma(x))$. □

[5] Commutativity is essential here, as we can see from the Weyl algebra example in (3.2)(7)!

Proposition 3.9. *Let A_k be a cancellable module, and N, N' be submodules of A. Suppose there exist $f,\, g \in \mathrm{End}_k(A)$ that restrict to mutually inverse isomorphisms $N \to N'$ and $N' \to N$. Then $A/N \cong A/N'$.*

Proof. Using the inclusion map $N \hookrightarrow A$ and the injective map $f : N \to A$, we can construct a pushout

$$P := (A \oplus A)/\{(s,\, -f(s) :\, s \in N\}.$$

Since f is defined on A, the Lemma above applies to give $P \cong A \oplus (A/N)$. Similarly, the pushout

$$Q := (A \oplus A)/\{(t,\, -g(t) :\, t \in N'\}$$

constructed from g satisfies $Q \cong A \oplus (A/N')$. Now consider the automorphism $\sigma : A \oplus A \to A \oplus A$ defined by $\sigma(a,b) = (b,a)$. Noting that

$$\sigma(s,\, -f(s)) = (-f(s),\, s) = (-f(s),\, -g(-f(s)) \quad (\forall\ s \in N),$$

we see that σ induces an isomorphism from P onto Q. Therefore, we have $A \oplus (A/N) \cong A \oplus (A/N')$. Since A is cancellable, it follows that $A/N \cong A/N'$. $\square$

§4. The Substitution Property

To complete the conceptual framework of cancellation and internal cancellation, we introduce one more notion — that of substitution, due to P. Crawley and L. Fuchs.

Definition 4.1. A k-module A is said to have the *substitution property* if, whenever B and C are direct summands in a module M_k each with a complement $\cong A$, they have a *common* complement A_0 (necessarily $\cong A$).

If A satisfies substitution, then, given $M = A \oplus B = A' \oplus C$ with $A' \cong A$, we can take a common complement A_0 for B, C as above, and deduce that $B \cong M/A_0 \cong C$. Thus, we have the following hierarchy of module-theoretic properties on A:

(4.2) Substitution $\Longrightarrow$ Cancellation $\Longrightarrow$ Internal Cancellation $\Longrightarrow$ Dedekind-Finite.

The last two implications are easily seen to be irreversible; we'll show in (4.7) below that the same is true for the first one. Here, let us first prove the following analogue of (3.3) for the substitution property. (This result will prove to be quite useful later!)

Proposition 4.3. *A direct sum of modules $A \oplus D$ has the substitution property iff A and D both do.*

Proof. Suppose $A \oplus D$ has the substitution property. To see that A does, consider a module $M = A \oplus B = A' \oplus C$, where $A' \cong A$. In $D \oplus M$, the submodules B and C have complements isomorphic to $A \oplus D$, so they have a common complement X. Then $X \cap M$ is a common complement for B and for C in M.

Conversely, suppose both A and D have the substitution property. To check that $A \oplus D$ also does, consider a module $N = (A \oplus D) \oplus B = (A' \oplus D') \oplus C$, where $A' \cong A$

and $D' \cong D$. Then $D \oplus B$ and $D' \oplus C$ have a common complement A_0 in N. But then $A_0 \oplus B$ and $A_0 \oplus C$ must have a common complement D_0 in N. Now $A_0 \oplus D_0$ gives a common complement for B and for C in N! $\square$

We come now to the important link between substitution and stable range 1. Here, we do not assume the result (1.8); in fact, the proof below will lead to a second, more conceptual proof of (1.8).

Theorem 4.4. (Warfield [Wa$_4$], Fuchs [Fu$_2$]) *A k-module A has the substitution property iff $R = \mathrm{End}(A_k)$ has right (resp. left) stable range* 1.

Proof. Assume that $\mathrm{sr}_r(R) = 1$, and let $M = A \oplus B = A' \oplus C$ with $A' \cong A$. *Fixing an identification of A' with A*, the projection of M onto A' (with kernel C) gives an exact sequence

$$(4.5) \qquad 0 \longrightarrow C \longrightarrow A \oplus B \xrightarrow{(f,g)} A \longrightarrow 0,$$

where the notation (f, g) is with respect to the decomposition $M = A \oplus B$. The splitting sending A back to A' can likewise be expressed by a pair $\begin{pmatrix} f' \\ g' \end{pmatrix}$, where $f' : A \to A$, and $g' : A \to B$. Since $\mathrm{Id}_A = (f, g)\begin{pmatrix} f' \\ g' \end{pmatrix} = f\,f' + g\,g'$, the assumption that $\mathrm{sr}_r(R) = 1$ implies that $f + (g\,g')h = u \in \mathrm{U}(R)$ for some $h \in R$ (by the right analogue of (1.7)). Thus, we have $(f, g)\begin{pmatrix} 1 \\ g'h \end{pmatrix} = u$. Let $A_0 := \mathrm{im}\begin{pmatrix} 1 \\ g'h \end{pmatrix} \subseteq M$. Clearly, $M = A_0 \oplus C$, and it is also easy to see that $M = A_0 \oplus B$ (noting that A_0 is the "graph" of the function $g'h : A \to B$). Thus A_0 is a common complement for B and for C, as desired.

For the "only if" part, assume A has the substitution property, and consider an equation $f\,f' + g\,g' = \mathrm{Id}_A$ in R. Let A', B be copies of A. The map $\pi : M = A \oplus B \to A'$ given by (f, g) is split by $\begin{pmatrix} f' \\ g' \end{pmatrix}$. Since A is cancellable, $C := \ker(\pi)$ is another copy of A. Since B, C both have complements $\cong A$, they have a *common* complement, say A_0, in M. Take any isomorphism $\varphi : A' \to A_0$, and view it as a map from A' to $M = A \oplus B$, with representation (say) $\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}$. Since A_0 is a complement for B, we must have $f_1 \in \mathrm{U}(R)$. On the other hand, $u := \pi\,\varphi \in \mathrm{U}(R)$ (since A_0 is also a complement for C). From $u = f\,f_1 + g\,g_1$, it follows that $f + g(g_1 f_1^{-1}) \in \mathrm{U}(R)$, so we have shown that $\mathrm{sr}_r(R) = 1$.

It should now be "clear" that the same results can be gotten for the *left* stable range of R! All we have to do is to fix again an identification of A' with A, and work with the split exact sequence

$$(4.5)' \qquad 0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow C \longrightarrow 0$$

given by projection of $M = A' \oplus C$ onto C, with kernel A' (identified with A). The same analysis as above now shows that A has the substitution property iff $\mathrm{sr}_\ell(R) = 1$.

Since *any* ring R can be represented as an endomorphism ring (e.g. $\mathrm{End}(R_R)$), we have arrived at another proof of (1.8). $\square$

Remark 4.6. Yet another piece of extra information given by the proof of (4.4) is the following: to check that a module A satisfies substitution, we need only check the condition in (4.1) in the special case where $B \cong C \cong A$.

Example 4.7. Taking $k = \mathbb{Z}$, we know by (3.6) that the k-module $\mathbb{Z}$ is cancellable. However, it has endomorphism ring $\mathbb{Z}$, which does not have stable range 1 (by (1.6)(2)), so (4.4) implies that $\mathbb{Z}$ *does not satisfy substitution.* From this, we see that the first implication in (4.2) is in general irreversible.

Of course, the fact that $\mathbb{Z}$ lacks substitution can be seen directly (without (4.4)), by crunching some numbers (cf. [W]). In $\mathbb{Z}^2$, let

$$(4.8) \qquad A = \mathbb{Z}\cdot(1,0), \quad B = \mathbb{Z}\cdot(0,1); \quad A' = \mathbb{Z}\cdot(7,3), \quad \text{and} \quad C = \mathbb{Z}\cdot(5,2).$$

Each of these is isomorphic to $\mathbb{Z}$, and we have $\mathbb{Z}^2 = A \oplus B = A' \oplus C$. *We claim that B and C do not have a common complement.* For, a common complement would have the form $\mathbb{Z}\cdot(a,b)$, for some a, $b \in \mathbb{Z}$. But then the two matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ 5 & 2 \end{pmatrix}$ must have determinant ± 1; that is, $a = \pm 1$, and $2a - 5b = \pm 1$. The former obviously contradicts the latter. (The same constructions in (4.8) will be used later in (6.8)(2) to show that the module $A \cong \mathbb{Z}$ does not satisfy "2-exchange".)

§5. Cancellation Theorems and Morita Invariance

In the first half of this section, we'll use the substitution property as a tool to obtain some concrete cancellation results. The first "obvious" application of (4.4) is the following.

Evans' Cancellation Theorem 5.1. [Ev] *If the endomorphism ring R of a module A_k has stable range* 1 *(e.g. R is semilocal), then A_k is cancellable.*

Proof. By (4.4), A_k has the substitution property. Thus, A_k is cancellable by (4.2). $\square$

The reason the proof of (5.1) is so short is that the main work for its proof was already all done in (4.4)! Perhaps a remark on the chronology of these results is in order. It was E. G. Evans who first proved (5.1) in 1973; the more general result (4.4) (with the stronger conclusion of "substitution") was obtained partly by Fuchs in 1971, and fully by Warfield in 1980.

An immediate class of modules to which (4.4) and (5.1) apply is the class of *strongly indecomposable modules*; that is, modules A_k for which $\mathrm{End}_k(A)$ is local. (Local rings have stable range 1 by (1.5)(2).) This includes all simple modules, but also some non-simple ones, such as $\mathbb{Q}$ as a $\mathbb{Z}$-module, or $\mathbb{Q}^2$ as a right module over $\begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$, etc. More generally, taking into account (3.3) and (4.3), we have the following.

Corollary 5.2. *Suppose a module A_k has a finite Azumaya decomposition, in the sense that A can be written as $A_1 \oplus \cdots \oplus A_n$, where $n < \infty$ and each A_i is strongly indecomposable. Then A has the substitution and cancellation properties.*

The Corollary applies, for instance, to all modules A of finite length (see [La$_1$: (19.17)]). To show further applications of (4.4) and (5.1), we record a few more of their interesting consequences, as follows.

Corollary 5.3. *A f.g. (right) R-module A has the substitution and cancellation properties if R is a module-finite algebra over a commutative noetherian semilocal ring k.*

Proof. Since A_k is (obviously) f.g. and k is noetherian, $\mathrm{End}_k(A)$ is f.g. as a k-module. It follows that $\mathrm{End}_R(A) \subseteq \mathrm{End}_k(A)$ is also f.g. as a k-module. From this, it is easy to see that $\mathrm{End}_R(A)$ is a semilocal ring [La$_1$: (20.6)]. Therefore, (4.4) and (5.1) again apply. $\square$

The following special case of (5.3) is often useful in integral representation theory: *if R is a module-finite algebra over a commutative noetherian ring k, then for any right R-modules A, B, C where A is f.g., $A \oplus B \cong A \oplus C$ implies that B and C belong to the same genus;* that is, $B_{\mathfrak{p}} \cong C_{\mathfrak{p}}$ as $R_{\mathfrak{p}}$-modules for any prime ideal $\mathfrak{p}$ of k. In the case where R itself is commutative and noetherian, results of this nature were already obtained by Vasconcelos as early as 1965; see [V]. In fact, in [V], Vasconcelos proved the cancellation theorem (5.1) under the stronger assumption that the module A in question has a *local* endomorphism ring.

Corollary 5.4. *Let R be a ring with* $\mathrm{sr}(R) = 1$, *and let P_R be a f.g. projective R-module. Then*

(1) *P has the substitution and cancellation properties;*

(2) $\mathrm{sr}(\mathrm{End}(P_R)) = 1$; *in particular,* $\mathrm{sr}(\mathbb{M}_n(R)) = 1$ *for any $n \geq 1$;*

(3) (Vaserstein [Va$_2$]) *for any idempotent $e \in R$, we have* $\mathrm{sr}(eRe) = 1$.

Proof. (1) In view of (4.3), it suffices to prove this for the free module R_R. This case follows from (4.4), since $\mathrm{End}(R_R) \cong R$ has stable range 1.

The rest is also easy: (2) follows from (1) and (4.4), and (3) follows from (2) since eR is a f.g. projective R-module with $\mathrm{End}_R(eR) \cong eRe$ [La$_1$: (21.7)]. $\square$

We note, in passing, that part (3) above can also be proved without using cancellation techniques. In fact, in [Va$_2$], (3) was first proved by a completely *ad hoc* stable range calculation. More recently, yet another approach to (5.4)(3) is given in [LD].

Using (5.4), we can now formulate the following refinement (and extension) of (2.6)(1) and (2.9).

Corollary 5.5. *For any regular ring R, the following are equivalent:*

(1) *R is unit-regular*;

(2) $\mathrm{sr}(R) = 1$;

(3) *any f.g. projective right* R*-module is cancellable*;

(4) *the category of f.g. projective right* R*-modules satisfies cancellation.*

If indeed R *is unit-regular, then*

(5) *for any f.g. projective module* P_R*,* $\mathrm{End}(P_R)$ *is also unit-regular; in particular, any matrix ring* $\mathbb{M}_n(R)$ *is unit-regular;*

(6) *for any idempotent* $e \in R$*, the corner ring* eRe *is unit-regular.*

Proof. (3) $\Rightarrow$ (4) is a tautology. (4) $\Rightarrow$ (1) follows from (2.6)(1), and (1) $\Rightarrow$ (2) was (2.9). Finally, (2) $\Rightarrow$ (3) follows from (5.4) (even without R being regular).

Since R is a regular ring, it is easy to show that, in (5), (6) above, the rings $\mathrm{End}(P_R)$ and eRe are also regular [Go$_3$: (1.7)]. Thus, combining (5.4) with (2) $\Leftrightarrow$ (1) above, we get (5) and (6). □

It is also possible to prove part (6) above without using cancellation techniques; for such a proof, see [LM].

The results above can be used now to show that the notion of stable range 1 is "Morita-invariant". Recall that a ring-theoretic property $\mathcal{P}$ is called Morita-invariant if, whenever a ring R has the property $\mathcal{P}$, then any ring S Morita-equivalent to R also does.[6] In studying the Morita invariance of the property $\mathrm{sr}(R) = 1$, the beginning point is to observe that the module-theoretic notions of substitution, cancellation, and internal cancellation are all of a *categorical* nature, in that they are defined purely in terms of Mod-R (the category of right R-modules), using categorical notions such as subobjects, direct sums, and isomorphisms. From this observation, it follows that, if two modules correspond to each other under a category equivalence, then one satisfies any of the properties in (4.2) iff the other does. This puts us in a good position to prove our desired result.

Theorem 5.6. *"Stable range* 1*" is a Morita-invariant property of rings.*

Proof. Fix a category equivalence $T:$ Mod-$S \to$ Mod-R, where S is any ring Morita-equivalent to R, where $\mathrm{sr}(R) = 1$. Then $P := T(S)$ is a f.g. projective module in Mod-R [La$_3$: (18.10)(A)]. By (5.4), P_R has the substitution property. By what we said before (5.6), therefore, the same is true for S_S. But then $S \cong \mathrm{End}(S_S)$ has stable range 1 by (4.4), as desired. □

The above theorem is all the more remarkable in view of the fact that, for $n > 1$, the property "stable range $= n$" is no longer Morita-invariant! The quickest way to see this is to quote the beautiful formula of Vaserstein on the stable range of a matrix ring.

Theorem 5.7. ([Va$_1$: Th. 2], [Wa$_4$: (1.12)]) *If a ring* k *has stable range* n*, then the matrix ring* $R = \mathbb{M}_m(k)$ *has stable range* $1 + \lceil \frac{n-1}{m} \rceil$*, where* $\lceil x \rceil$ *(the ceiling function on* x*) denotes the smallest integer* $\geq x$*.*

[6] We say that S is Morita-equivalent to R if the module categories over R and S are equivalent.

Recall that the matrix ring $\mathbb{M}_m(k)$ is always Morita-equivalent to k. If $\mathrm{sr}(k) = 1$, (5.7) shows that $\mathrm{sr}(\mathbb{M}_m(k)) = 1$ for all m: this is, of course, already predicted by (5.6). However, let us consider a ring k with $\mathrm{sr}(k) = n \geq 2$. For any $m \geq n-1$, (5.7) shows that $\mathrm{sr}(\mathbb{M}_m(k)) = 2$. Thus, the property "stable range $= n$" ($n \geq 2$) is *not* Morita-invariant. In fact, the above construction shows that, for rings R of stable range ≥ 2, a Peirce corner ring eRe (even for full idempotents e) may have arbitrarily large stable range!

To give a somewhat different application of (5.1), let us conclude this section with the following concrete cancellation result (and its interesting corollary).

Theorem 5.8. (Hsü) *Any f.g. module A over a Dedekind domain k is cancellable.*

Proof. By the structure theorem on f.g. modules over k, A is a finite direct sum of copies of I and k/J, where I, J are ideals of k. Since I is f.g. projective, it is cancellable by (3.3) (as k itself is cancellable, by (3.6)). On the other hand, if $A = k/J$ where J is a nonzero ideal of k, then $\mathrm{End}_k(A) \cong A$ as rings, and A is a *semilocal* ring. In this case, A is cancellable by (5.1). $\square$

The above result was first proved (independently) over $\mathbb{Z}$ by Cohn [Co] and by Walker [W], and later extended to general Dedekind domains by Hsü [Hs]. Following Hsü, we record the following interesting application of (5.8) to another type of cancellation problem, called "n-cancellation" in [La$_4$]. For any module C, let us write $n \cdot C$ for the direct sum of n copies of C.

Corollary 5.9. *Let A, B be f.g. torsion modules over a Dedekind domain k. If $n \cdot A \cong n \cdot B$ (for some $n \geq 1$), then $A \cong B$.*

Proof. By the Invariant Factor Theorem over k [CR: (22.12)], A is isomorphic to $(k/I_1) \oplus \cdots \oplus (k/I_r)$, where $I_1 \subseteq \cdots \subseteq I_r$ are suitable ideals in k. Here, I_1 is the "order ideal" of A. Thus, we can write $A \cong (k/I_1) \oplus A'$ for some A'. Since $n \cdot A \cong n \cdot B$, B also has order ideal I_1, so we can write $B \cong (k/I_1) \oplus B'$, for some B'. From

$$n \cdot (k/I_1) \oplus n \cdot A' \cong n \cdot (k/I_1) \oplus n \cdot B',$$

we get $n \cdot A' \cong n \cdot B'$ by cancelling $n \cdot (k/I_1)$ (using (5.8)). Since A', B' are still f.g. and torsion, we may assume by induction that $A' \cong B'$, and hence $A \cong B$. $\square$

It is of interest to ask if the result (5.9) also holds for f.g. *torsionfree* modules over a Dedekind domain k. The answer is "no" in general: a simple application of Steinitz' Isomorphism Theorem shows readily that *f.g. torsionfree k-modules satisfy n-cancellation (as in* (5.9)*) if and only if the class group of k has no n-torsion.* For this statement, however, the "f.g." assumption on the modules is essential. Even for the ring of rational integers $k = \mathbb{Z}$ (which has a trivial class group), it can be shown that *non-f.g.* torsionfree modules fail to satisfy n-cancellation. For instance, using the same technique as in (3.2)(5), B. Jónsson has constructed an example of rank 2 torsionfree abelian groups

A, B such that $A \oplus A \cong B \oplus B$, but $A \not\cong B$; see [Jó: §3]. Historically, this provided a negative solution to Kaplansky's "Second Test Problem" for infinite abelian groups raised in §6 of his book [Ka_1].

§6. The Exchange Property

There is another very useful module-theoretic property, called the *exchange property*, which is a byproduct of the general investigation by Crawley and Jónsson [CJ] on the decompositions of algebraic systems (see also [JT]). In the last two decades, modules with the exchange property have become quite a popular topic for research. Since the notion of exchange has a large bearing on the material in §§3-5, we shall introduce its definition here, and develop a few central results on exchange modules and their relations to cancellation and substitution problems in this section.

Definition 6.1. A module A_k over a ring k is said to have the *n-exchange property* (or A is an *n-exchange module*) if, whenever (a copy of) A is a direct summand in any module $M = M_1 \oplus \cdots \oplus M_n$, A has a complement in M of the form $M_1' \oplus \cdots \oplus M_n'$ for suitable submodules $M_i' \subseteq M_i$.

Remark 6.2. In the above situation, each M_i' is a direct summand of M, and hence of M_i. Thus, we can write $M_i = M_i' \oplus M_i''$ for suitable submodules $M_i'' \subseteq M_i$. This observation about the M_i's in (6.1) will prove to be handy in dealing with n-exchange modules.

In Def. (6.1), we did not say what n was. In the most general case, we can take n to be any cardinal number. If A satisfies n-exchange for all cardinals n (resp. all finite cardinals n), A is said to have the *exchange property* (resp. the *finite exchange property*); more informally, we speak of A as an *exchange module* or a *finite exchange module*, respectively.

In this section, we shall be primarily interested in the case of finite exchange, so let us assume for now that $n < \infty$. We may ignore the case $n = 1$, since 1-exchange is a vacuous property. From Def. (6.1), it is clear that n-exchange implies r-exchange if $r \le n$. It turns out that the converse is also true if $r \ge 2$. In particular, *for (finite) $r \ge 2$, the r-exchange properties are all equivalent to* 2*-exchange.* This follows by an easy induction; the detailed proof is available, for instance, in [Fa_2: (2.5)].

As it also turns out, in checking the n-exchange property for a module A_k, it is sufficient to check the condition in (6.1) in a special case, as follows. (This result will be useful in §7.)

Proposition 6.3. *Suppose A_k satisfies the condition in* (6.1) *in the special case where each module M_i is embeddable in A. Then A satisfies n-exchange.*[7]

[7]The strongest result in this direction is that even the case $M_i \cong A$ for all i would suffice. For a proof of this, see [ZZ].

Proof. Suppose $M = A \oplus N = M_1 \oplus \cdots \oplus M_n$. Let $N_i := N \cap M_i$ (for all i), and let us write "bar" for the projection map from M to $M/(N_1 \oplus \cdots \oplus N_n)$. Then,

$$\overline{M} = \overline{A} \oplus \overline{N} = \overline{M}_1 \oplus \cdots \oplus \overline{M}_n.$$

Here, $A \cong \overline{A}$, and each $\overline{M}_i \cong M_i/N_i$ is embeddable in A (the projection $M \to A$ with kernel N induces the required embeddings). Thus, by the given hypothesis, we can write

$$\overline{M} = \overline{A} \oplus \overline{M_1'} \oplus \cdots \oplus \overline{M_n'}\,, \tag{6.4}$$

for suitable submodules $\overline{M_i'} \subseteq \overline{M}_i$. Letting M_i' be the preimage of $\overline{M_i'}$ in M_i, we can check readily that $M = A \oplus M_1' \oplus \cdots \oplus M_n'$, as desired. (Clearly, throughout this proof, n can be any cardinal number.) $\square$

It is perhaps not surprising that, for exchange modules, we have the following analogue of (3.3) and (4.3), due to Crawley and Jónsson.

Proposition 6.5. *A module $A \oplus D$ satisfies n-exchange (where n is any cardinal) iff A and D both do.*

Proof. The "if" part is not needed below, so we shall skip it. The "only if" part is proved easily by starting with an equation $A \oplus N = M_1 \oplus \cdots \oplus M_n$, and "adding" the module D; the details of the proof will be left to the reader. $\square$

In the case of *indecomposable* modules A, one can give an explicit characterization of the 2-exchange property for A in terms of a more familiar property. The precise statement, due to Warfield [Wa_1], is as follows.

Theorem 6.6. *Let A_k be an indecomposable k-module. Then A satisfies 2-exchange (or equivalently, finite exchange) iff A is strongly indecomposable. In particular, an indecomposable 2-exchange module has the substitution property (and hence also the three other properties in (4.2)).*

Proof. First suppose A satisfies 2-exchange. If $R = \mathrm{End}_k(A)$ is not local, we'll have $\alpha - \beta \in \mathrm{U}(R)$ for some nonunits $\alpha, \beta \in R$. Let $M = M_1 \oplus M_2$, where M_1, M_2 are copies of A. Then $(\alpha, \beta) : A \to M$ is a split monomorphism. Let A' be its image in M. Using 2-exchange on $A' \cong A$, we have thus $M = A' \oplus X \oplus Y$, with, say, $M_1 = X \oplus X'$ and $M_2 = Y \oplus Y'$ (see (6.2)). Then $A' = M/(X \oplus Y) \cong X' \oplus Y'$. Since A' is indecomposable, we may assume, say, $Y' = 0$, so $M = A' \oplus X \oplus M_2$. Clearly, $X \neq M_1$, so we must have $X = 0$ (since M_1 is indecomposable), and hence $M = A' \oplus M_2$. Thus, the projection π_1 of $M = M_1 \oplus M_2$ onto M_1 must map A' isomorphically onto M_1. But the composition $\pi_1 \circ (\alpha, \beta)$ is just the map α, and this is *not* an isomorphism: a contradiction.

For the converse, we assume R is local. Starting with a module of the form $M = A \oplus B = M_1 \oplus M_2$, we'll try to do a 2-exchange for A. Here, we use the basic idea of

Azumaya's proof of the Krull-Schmidt-Azumaya Theorem. Let π_i be the projection(s) of M to M_i. By the first step of Azumaya's proof (see [La$_1$: (19.21)]), the fact that $R = \mathrm{End}_k(A)$ is local enables us to assume that, say $\pi_1 : A \to M_1$ is a split monomorphism. Let $H = \pi_1(A)$, and write $M_1 = H \oplus K$. We claim that

$$M = A \oplus K \oplus M_2. \tag{6.7}$$

If this holds, then we will have accomplished our 2-exchange (since $K \subseteq M_1$). To check that the sum in (6.7) is direct, let $a + x + y = 0$, where $a \in A$, $x \in K$, and $y \in M_2$. Applying π_1, we get $\pi_1(a) + x = 0$, so $x = a = y = 0$. It thus only remains to show that $H \subseteq A + M_2$, for then $A + K + M_2$ contains H, K, as well as M_2, so it will be equal to M, proving (6.7). To see that $H \subseteq A + M_2$, consider any $h \in H$, say $h = \pi_1(a)$, where $a \in A$. Then $\pi_1(h - a) = h - \pi_1(a) = 0$ yields $h - a \in M_2$, which shows that $h \in A + M_2$, as desired. □

Remark 6.8. (1) In [CJ], Crawley and Jónsson have shown that, for the module A in (6.6), the 2-exchange (or finite exchange) property is equivalent to the general exchange property. This equivalence also holds, of course, if A is any *finitely generated* module.[8] However, whether this equivalence holds for *all* modules has remained a difficult open question in the study of the exchange theory of modules. For more information on this issue, see (8.10) and (E) in §9 below.

(2) It follows from (6.6) that, over the ring $k = \mathbb{Z}$, the module $\mathbb{Z}$ does not satisfy 2-exchange. (This module is indecomposable but not strongly indecomposable.) This can also be seen directly as follows. Using the decompositions $\mathbb{Z}^2 = A \oplus B = A' \oplus C$ in (4.8), we claim that no exchange is possible for $A = \mathbb{Z} \cdot (1, 0)$ with respect to the decomposition $\mathbb{Z}^2 = A' \oplus C$. Indeed, if $\mathbb{Z}^2 = A \oplus X \oplus Y$ where $X \subseteq A'$ and $Y \subseteq C$, then in view of (6.2), X must be (0) or A', and Y must be (0) or C. But then $\mathbb{Z}^2 = A \oplus A'$, or $\mathbb{Z}^2 = A \oplus C$. This is impossible, since neither $\{(1, 0),\, (7, 3)\}$ nor $\{(1, 0),\, (5, 2)\}$ is a basis for $\mathbb{Z}^2$. This shows that the $\mathbb{Z}$-module $A \cong \mathbb{Z}$ does not satisfy 2-exchange.

From (6.6), we can show that, in many cases, a module satisfying 2-exchange must have the substitution property, as follows.

Corollary 6.9. *Let A be a finite direct sum of indecomposable modules. If A satisfies 2-exchange, then A has the substitution property (and hence also the three other properties in (4.2)).*

Proof. Say $A = A_1 \oplus \cdots \oplus A_r$, where $r < \infty$ and each A_i is indecomposable. If A satisfies 2-exchange, then so does A_i (by the "only if" part of (6.5)), and hence A_i has the substitution property by (6.6). Now (4.3) implies that A itself has the substitution property. □

[8] If A is f.g., then in Def. (6.1), A is already a direct summand in the direct sum of a *finite number* of the M_i's. From this, it is clear that the finite exchange property on A implies the general exchange property.

Remark 6.10. Of course, the converse of (6.9) is not true. For instance, if A is the underlying abelian group of the ring $R = \mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$ where p, q are distinct primes, then $\mathrm{End}(A_{\mathbb{Z}})$ is the semilocal ring R. This endomorphism ring has stable range 1 by (2.10), so A has the substitution property by (4.4). However, R is not local, so $A_{\mathbb{Z}}$ is (indecomposable but) not strongly indecomposable. Thus, (6.6) implies that A does not satisfy 2-exchange.

We should also note that, without the initial assumption on A in (6.9) (namely, that it is a finite direct sum of indecomposables), the 2-exchange property on A need not imply any of the properties in (4.2).[9] Nevertheless, it turns out that, in the presence of 2-exchange, the first three properties in the display (4.2) are equivalent to one another. This is a result due to H.-P. Yu. We shall be able to prove, however, only a part of this theorem, as the reader will see below.

Theorem 6.11. [Yu] *Let A be a module with the 2-exchange (or equivalently, finite exchange) property. Then the following conditions on A are equivalent:*

(1) *A is internally cancellable;*
(2) *A is cancellable;*
(3) *A has the substitution property.*

Proof. According to (4.2), the implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are true for *all* modules A. Thus, the job here is to prove the key implication (1) $\Rightarrow$ (3) for a module A satisfying 2-exchange. Yu's proof of this (in [Yu: p. 109]) makes use of certain results on finite exchange modules and their endomorphism rings due to Warfield and Nicholson. Since we do not (yet) have these results at our disposal, we shall make a compromise here and only prove (1) $\Rightarrow$ (2) below (thus getting the equivalence of cancellation and internal cancellation). This implication was, in fact, proved much earlier by Fuchs in [Fu$_3$]. The more than twenty-year time gap between the proofs of (1) $\Rightarrow$ (2) and the harder (1) $\Rightarrow$ (3) is perhaps a good indication of the relative depth of the latter implication.

Assume that A is internally cancellable, and consider a module $M = A \oplus B = A' \oplus C$, where $A' \cong A$. Since A is assumed to have 2-exchange, we can write $M = A' \oplus X \oplus Y$ for suitable submodules $X \subseteq A$ and $Y \subseteq B$. Write $A = U \oplus X$ and $B = V \oplus Y$ (see (6.2)). Then

$$A' \cong \frac{M}{X \oplus Y} = \frac{A \oplus B}{X \oplus Y} \cong U \oplus V.$$

Since $A' \cong A = U \oplus X$ has internal cancellation, we have $X \cong V$. Therefore, $B = V \oplus Y \cong X \oplus Y \cong M/A' \cong C$, so we have proved the cancellation property for A. $\square$

While the proof for (1) $\Rightarrow$ (2) above was good enough to show that $B \cong C$, it seemed to offer no clue toward proving that B and C have a common complement in M. Thus, at least before we know any better, some knowledge of the results of Warfield [Wa$_3$] and Nicholson [Ni] seems to be necessary for the full proof of (6.11). These results

[9] To see this, consider the ring R constructed in (2.11). For this ring, R_R is not Dedekind-finite. But R is von Neumann regular, so according to (F) in §9 below, R_R is an exchange module.

will be summarized below toward the end of §8. At that point, we will also return to give a sketch for Yu's proof of (1) $\Rightarrow$ (3) in (6.11).

We close by pointing out why Dedekind-finiteness cannot be added to the list of equivalent properties in (6.11). It is known that there exists a von Neumann regular ring R that is Dedekind-finite, but not unit-regular: see [Go_3: (5.10)]. By Footnote 9, R_R is an exchange module, but R_R does not satisfy any of the (equivalent) properties in (6.11).

§7. Quasi-Injective Modules

In this section, we shall study a subclass of exchange modules consisting of the so-called quasi-injective modules. For this subclass, it turns out that Yu's theorem (6.11) can be further refined. Quasi-injective modules are defined by a weakening of the well-known notion of injectivity, as follows.

Definition 7.1. A module A_k over a ring k is called *quasi-injective* if, for any submodule $B \subseteq A$, any $f \in \mathrm{Hom}_k(B, A)$ can be extended to an endomorphism of A.

If A_k is an *injective* module, then for any two modules $B \subseteq C$, any $f \in \mathrm{Hom}_k(B, A)$ can be extended to some $g \in \mathrm{Hom}_k(C, A)$. Since we can, in particular, take C to be A, we see that an injective module A is always quasi-injective. The converse of this is not true: for instance, any semisimple module is (clearly) quasi-injective, but it need not be injective. The following proposition gives another interesting family of quasi-injective modules which need not be injective.

Proposition 7.2. *Let k be a commutative principal ideal domain, and let I be a nonzero ideal in k. Then the cyclic k-module $A = k/I$ is quasi-injective (but not necessarily injective).*

Proof. Let f be any k-homomorphism from a submodule $J/I \subseteq A$ to A. Since k is a PID, we have $I = ak$ and $J = bk$ for some $a, b \in k$, say with $a = bc$. We have $f(b + I) = x + I$ for some $x \in k$. Then

$$0 = f(a + I) = f(bc + I) = xc + I$$

shows that $xc = ay$ for some $y \in k$. Since $a = bc \neq 0$, we have $x = by$, so f is the restriction of the endomorphism of A given by the multiplication by y. This checks that A is quasi-injective. The example $k = \mathbb{Z}$ with $I = n\mathbb{Z}$ $(n \neq 0)$ shows that in general $A = k/I$ need not be injective. $\square$

The next result gives a powerful characterization of a quasi-injective module in terms of the endomorphisms of its injective hull.

Theorem 7.3. *A module A (over any ring) is quasi-injective iff it is fully invariant in its injective hull $E := E(A)$ (that is, iff any endomorphism of E takes A into A).*

Proof. The sufficiency part is easy. Given a k-homomorphism $f : B \to A \subseteq E$ where $B \subseteq A$, we can extend f to some $g : E \to E$ (since E is injective). By assumption, we have $g(A) \subseteq A$. Thus, $g : A \to A$ extends the given f, which checks the quasi-injectivity of A.

Conversely, assume A is quasi-injective, and consider any $f \in \mathrm{End}_k(E)$. The set $B := \{a \in A : f(a) \in A\}$ is easily checked to be a k-submodule of A. Since f maps B into A, there exists $g \in \mathrm{End}_k(A)$ such that $f|_B = g|_B$. We may assume that $g \in \mathrm{End}_k(E)$, thanks to the injectivity of E. Assume, for the moment, that $(g-f)A \neq 0$. Then $(g-f)A \cap A \neq 0$ (since A is essential in E), so $(g-f)a = a' \neq 0$ for some $a, a' \in A$. Then $f(a) = g(a) - a' \in A$ implies that $a \in B$. But $f|_B = g|_B$ shows that $f(a) = g(a)$, and so $a' = 0$, a contradiction. Therefore, we must have $(g-f)A = 0$, and so $f(A) = g(A) \subseteq A$, as desired. ☐

We shall record for later use four nice consequences of Theorem 7.3.

Corollary 7.4. *If a direct sum $A \oplus D$ is quasi-injective, then so are A and D.*

Proof. Let $f \in \mathrm{End}(E(A))$. Then $f \oplus \mathrm{Id}_{E(D)}$, as an endomorphism of $E(A) \oplus E(D) \cong E(A \oplus D)$, takes $A \oplus D$ into itself. But then f must take A into A, so another application of (7.3) shows that A (and similarly, D) is quasi-injective.[10] ☐

Corollary 7.5. *If A is a quasi-injective module, then so is A^n (the direct sum of n copies of A) for any $n < \infty$.*

Proof. Upon identifying $E(A^n)$ with $E(A)^n$, every endomorphism of $E(A^n)$ is represented by a matrix in $f = (f_{ij})$, where $f_{ij} : E(A) \to E(A)$. Since $f_{ij}(A) \subseteq A$ for all i, j, it follows that $f(A^n) \subseteq A^n$. Thus, (7.3) implies that A^n is quasi-injective. ☐

Corollary 7.6. *Let A be a quasi-injective module, with injective hull E. If $Q \subseteq E$ is isomorphic to a subquotient of A, then we must have $Q \subseteq A$.*

Proof. The subquotient assumption means that there exists a surjective homomorphism f from a submodule $B \subseteq A$ onto Q. Viewing f as from B to E, we may extend it to an endomorphism $g : E \to E$. By (7.3), we have $g(A) \subseteq A$, so we have $Q = f(B) = g(B) \subseteq A$. ☐

Corollary 7.7. *Let A be a quasi-injective module, with injective hull E. Then A is Dedekind-finite iff E is.*

Proof. First assume E is Dedekind-finite. Suppose $A = A' \oplus X$, where $A' \cong A$. Taking injective hulls, we have

$$E \cong E(A') \oplus E(X) \cong E(A) \oplus E(X) \cong E \oplus E(X).$$

[10] Of course, a direct proof of (7.4) using Def. (7.1) is possible. The present proof is chosen mainly to illustrate the use of (7.3)! Also, we note in passing that, in general, the converse of (7.4) is not true; see [La$_3$: (6.73)(2)].

Thus, $E(X) = 0$, and so $X = 0$.

Conversely, assume A is Dedekind-finite, and suppose $E = E' \oplus Y$, where $E' \cong E$. Let $g \in \operatorname{End}(E)$ be such that it maps E isomorphically onto E', and let $f \in \operatorname{End}(E)$ be such that $f : E' \to E$ is the inverse of $g : E \to E'$, and $f(Y) = 0$. Then $fg = 1_E$. By (7.3), $f_0 := f|_A$ and $g_0 := g|_A$ are endomorphisms of A, also with $f_0 g_0 = 1_A$. Thus,

$$A = \operatorname{im}(g_0) \oplus \ker(f_0), \quad \text{where } \operatorname{im}(g_0) \cong A,$$

by the injectivity of g_0. Since A is Dedekind-finite, we must have $0 = \ker(f_0) = \ker(f) \cap A$. But A is essential in E, so $Y = \ker(f) = 0$, as desired.[11] □

We come now to the main theorem that gives the basic connection between injectivity and exchange. This result was first proved for injective modules by Warfield, and then for quasi-injective modules by Fuchs.

Theorem 7.8. ([Wa$_2$], [Fu$_1$]) *Any quasi-injective module A is an exchange module.*

Proof. Thanks to (6.3), it suffices to check the exchange property of A by starting with an equation $M = A \oplus N = M_1 \oplus \cdots \oplus M_n$, where *each M_i is embeddable in A*. (Throughout this proof, n denotes an arbitrary cardinal number.) By Zorn's Lemma, there exists a submodule $X \subseteq M$ that is maximal with respect to the following two properties:

(1) X has the form $M_1' \oplus \cdots \oplus M_n'$, where $M_i' \subseteq M_i$ for all i;
(2) $X \cap A = 0$.

We are done if we can show that $M = A + X$, for then $X = M_1' \oplus \cdots \oplus M_n'$ will provide the desired type of complement to A. Let "bar" denote the projection map from M to M/X, and consider the submodule

$$\overline{A} \subseteq \overline{M} = \overline{M_1} \oplus \cdots \oplus \overline{M_n}\,.$$

Each intersection $\overline{A} \cap \overline{M_i}$ must be essential in $\overline{M_i}$ (for, if $\overline{M_i}$ has a nonzero submodule with zero intersection with $\overline{A}$, we would be able to "enlarge" the choice of M_i' to get a bigger X with the properties (1), (2) above). It follows (easily) that $\bigoplus_i (\overline{A} \cap \overline{M_i})$ is essential in $\bigoplus_i \overline{M_i} = \overline{M}$, so in particular, $\overline{A}$ itself is essential in $\overline{M}$. We may thus choose an injective hull of $\overline{A}$ that contains $\overline{M}$. But each $\overline{M_i}$ is isomorphic to a subquotient of $\overline{A}$ (since M_i is embeddable in $A \cong \overline{A}$), so (7.6) implies that $\overline{M_i} \subseteq \overline{A}$. Summing over i, we see that $\overline{M} \subseteq \overline{A}$, which gives the desired equation $M = A + X$. □

Corollary 7.9. *If a quasi-injective module A has a decomposition into a finite direct sum of indecomposable modules $A_1 \oplus \cdots \oplus A_n$, then this is an Azumaya decomposition, and such a decomposition of A is unique (up to isomorphism, and a permutation*

[11]The proof of the "if" part in this Corollary did not use the fact that A is quasi-injective. Thus, this part holds for any module A. However, the reader can check that the "only if" part need not hold for arbitrary modules.

of the summands). Furthermore, the module A has the substitution and cancellation properties.

Proof. By (7.4), each A_i is quasi-injective, so it is an exchange module by (7.8). Therefore, by (6.6), A_i is strongly indecomposable, so $A = A_1 \oplus \cdots \oplus A_n$ is automatically an Azumaya decomposition. The uniqueness of the decomposition of A now follows from the Krull-Schmidt-Azumaya Theorem [La_1: (19.21)]. The substitution and cancellation properties of A follow from (5.2). $\square$

Under reasonable finiteness conditions on A, it will have the type of decompositions required in (7.9). For instance, one of the more popular conditions is that A has a finite uniform dimension (that is, A does not contain an infinite direct sum of nonzero submodules). Under such a condition, the conclusions of (7.9) will apply to the quasi-injective module A. However, not all quasi-injective modules can be decomposed as in (7.9). Thus, it would be desirable to have a result on the cancellability of quasi-injective modules that does not presuppose such a decomposition. Such a result is, indeed, available, by a further specialization of Yu's theorem (6.11). In a nutshell, for quasi-injective modules, *the four properties in the diagram* (4.2) *all become equivalent*! This is, however, not easy to prove. Again, we will have to skip the hardest part of this result, and merely state it without proof, as follows.

Suzuki's Theorem 7.10. [Su] *Let A be a quasi-injective module. Then A is Dedekind-finite iff any isomorphism $f : N \to N'$ from one submodule of A to another extends to an automorphism of A. In particular, if A is Dedekind-finite, then for any two isomorphic submodules $N \cong N'$ in A, we have $A/N \cong A/N'$.*

The "if" part in the second sentence of (7.10) is clear, so the depth of (7.10) lies in the "only if" part. Here, we can certainly extend $f : N \to N'$ to an endomorphism of A (using the quasi-injectivity of A), but the theorem asserts the existence of an extension that is an *automorphism*! The last conclusion of this theorem reminds us of the result (3.9). However, (3.9) doesn't help here, since A was assumed to be cancellable in (3.9), but in (7.10), A is only assumed to be Dedekind-finite (and quasi-injective). Suzuki's proof (of the "only if" part in (7.10)) consists of a reduction to the case where A is injective, and ultimately depends on an argument involving a 1-sided self-injective Dedekind-finite von Neumann regular ring. We refer the reader to [Su] for the details.

Before stating the final theorem, we need to recall the notion of hopfian and cohopfian modules. Over any ring k, a module A_k is said to be *hopfian* if every surjective endomorphism of A is an automorphism, and *cohopfian* if every injective endomorphism of A is an automorphism. An easy argument shows that either one of these conditions implies that A is Dedekind-finite. With this remark, we can now state and prove the following result, which should be collectively attributed, in chronological order, to Suzuki [Su], Birkenmeier [Bi], and Yu [Yu]; see also Canfell [Ca: (3.5), (3.9)], where the equivalence of some of the conditions below is proved from the viewpoint of the completion of certain diagrams using module automorphisms.

Theorem 7.11. *For any quasi-injective module* A*, the following are equivalent:*

(0) A *is Dedekind-finite.*
(1) A *is internally cancellable.*
(2) A *is cancellable.*
(3) A *has the substitution property;*
(4) A *is cohopfian.*

Furthermore, each of these properties is implied by: (5) A *is hopfian.*

Proof. That each of (4) and (5) implies (0) was already pointed out in the paragraph preceding (7.11). (0) $\Rightarrow$ (1) is hard, but this follows from Suzuki's Theorem (7.10).[12] The equivalence of (1), (2), and (3) follows from (6.11), which is applicable here since the quasi-injective module A has the (general) exchange property, by (7.8). Thus, our only remaining job is to prove that (2) $\Rightarrow$ (4).

Assume that A is cancellable, and consider any injection $f : A \to A$. Then $B := f(A) \cong A$, so we can take an isomorphism $B \to A$, and extend it to an endomorphism $g : A \to A$ (using the quasi-injectivity of A). For $K = \ker(g)$, we then have $A = K \oplus B$. Since $B \cong A$, cancellation of A yields $K = 0$, so $B = A$. This proves that f is an automorphism, and hence A is cohopfian. □

Remark 7.12. (A) It is known that, in the above theorem, the quasi-injective assumption on A can be further weakened. In fact, (7.11) is known to be true for the wider class of *continuous modules*; for a proof of this, see Chapter 3 of [MM$_2$]. (Incidentally, [MM$_2$] is an excellent reference for much of the material in §§6-7 in this paper.) For *quasi-continuous* modules, (0) and (1) are still equivalent [MM$_2$: (2.33)]. However, they are no longer equivalent to (2), as one can see from examples such as (3.2)(3). The technical point here is that *continuous modules have the exchange property* [MM$_2$: (3.24)], *but quasi-continuous modules do not.*

(B) As for the "hopfian condition" (5). we should point out that, even for injective modules A, the conditions (0)–(4) in (7.11) are, in general, weaker than (5). For instance, over the ring $k = \mathbb{Z}$, the Prüfer p-group A (the p-primary component of the group $\mathbb{Q}/\mathbb{Z}$) is injective, and indecomposable (and hence Dedekind-finite and cohopfian), but the endomorphism of A given by multiplication by p is surjective and has kernel $\cong \mathbb{Z}/p\mathbb{Z}$, so A is not hopfian.

We finish by recording a number of consequences of (7.11).

Corollary 7.13. *Let* E *be the injective hull of a quasi-injective module* A*. Then the conditions* (0)–(4) *in* (7.11) *are also equivalent to the conditions* $(0)'$–$(4)'$ *obtained from them by replacing* A *by* E*.*

[12] It is worth noting that we do not need at all the full strength of Suzuki's Theorem. In fact, (0) $\Rightarrow$ (1) here is just the last part of Suzuki's Theorem — in the special case when N and N' are direct summands of A. There is also an alternative way to get (0) $\Rightarrow$ (1), by using the method of nonsingular injective modules. For a sketch of this approach, see [Go$_1$].

Proof. Replacing A by E and applying (7.11), we see that $(0)'$–$(4)'$ are equivalent. Finally, by (7.7), we have $(0) \Leftrightarrow (0)'$. □

Corollary 7.14. *If $M = A \oplus D$ is quasi-injective, then M is cohopfian (resp. Dedekind-finite) iff A and D are cohopfian (resp. Dedekind-finite).*

Proof. By (7.4), A and D are quasi-injective. Thus, we can apply (7.11) to each of them, and deduce the Corollary from (3.3). □

Corollary 7.15 *Over a commutative ring k, any finitely generated quasi-injective module A satisfies the conditions* (0)–(4) *in* (7.11), *and the conditions* $(0)'$–$(4)'$ *in* (7.13).

Proof. By a well-known result of Strooker and Vasconcelos, any f.g. module A over a commutative ring k is hopfian (see, e.g. [La$_2$: Ex. (20.9)]). It follows from (7.11) and (7.13) that, if A is, in addition, quasi-injective, then the conditions (0)–(4) and $(0)'$–$(4)'$ there all hold. □

Last but not least, we have the following special case of (7.11) when A is taken to be the right module R_R.

Corollary 7.16. *For a right self-injective ring R, the following are equivalent:*

(0) *R is Dedekind-finite.*
(1) *R_R is internally cancellable.*
(2) *R_R is cancellable.*
(3) $\mathrm{sr}(R) = 1$.
(4) *Every non left* 0*-divisor in R is a unit.*

In particular, if a regular ring R is right self-injective and Dedekind-finite, then R is a unit-regular ring.

In the theory of self-injective rings, it is known that a *left and right* self-injective ring is always Dedekind-finite; for a proof of this, see [La$_3$: (6.48), (6.49)]. Assuming this result, we have:

Corollary 7.17. *Any left and right self-injective ring has stable range* 1.

§8. Endomorphism Ring Properties

In this section, we'll give a short discussion on the nature of some of the module-theoretic properties that we have so far studied in these notes. It is hoped that this discussion will help further unify these properties, and also put them in a better perspective in the study of module-theoretic properties in general. Also, some of the viewpoints taken in this section do not seem to have been made explicit in the literature before.

Definition 8.1. Let $\mathcal{P}$ be a module-theoretic property. We say that $\mathcal{P}$ is an *endomorphism ring property* (or ER-property for short) if, whenever A_k and $A'_{k'}$ are right modules (over possibly different rings k and k') such that $\mathrm{End}(A_k) \cong \mathrm{End}(A'_{k'})$ (as rings), A_k satisfies $\mathcal{P}$ implies that $A'_{k'}$ does (and then, of course, also conversely).

The idea of $\mathcal{P}$ being an "ER- property" is that whether or not a module A_k satisfies $\mathcal{P}$ depends solely upon the knowledge of the endomorphism ring $\mathrm{End}(A_k)$. The following proposition offers a useful characterization of such ER-properties.

Proposition 8.2. *Let $\mathcal{P}$ be a module-theoretic property. Then $\mathcal{P}$ is an* ER*-property iff the following holds:*

$(*)$ *For any module A_k (over any ring k) with endomorphism ring $R := \mathrm{End}(A_k)$, A_k satisfies $\mathcal{P}$ iff R_R satisfies $\mathcal{P}$.*

Proof. First suppose $\mathcal{P}$ is an ER-property. For any module A_k with an endomorphism ring $R = \mathrm{End}(A_k)$, we have

$$\mathrm{End}\,(R_R) \cong R \cong \mathrm{End}\,(A_k) \quad \text{(as rings)}.$$

Therefore, by the definition of the ER-property, A_k satisfies $\mathcal{P}$ iff R_R does, which proves $(*)$.

Conversely, assume that $(*)$ holds for the property $\mathcal{P}$. Let A_k and $A'_{k'}$ be two modules (possibly over different rings k and k') with rings of endomorphisms both isomorphic to some ring R. Assume that A_k satisfies $\mathcal{P}$. Then, applying $(*)$ to A_k, we see that R_R satisfies $\mathcal{P}$. Applying $(*)$ again to $A'_{k'}$, we conclude that $A'_{k'}$ also satisfies $\mathcal{P}$. This shows that $\mathcal{P}$ is an ER-property, as desired. $\square$

ER-properties are rather special among module-theoretic properties. A few random choices from some of the more familiar module properties all turn out to be non-ER, as follows.

Examples 8.3. The module properties "cyclic", "f.g." (finitely generated), "free", "projective", "injective", "noetherian", "artinian", "finite length", "simple", "semisimple", etc. are all non-ER. In fact, consider the module $A = \mathbb{Q}$ over the ring $k = \mathbb{Z}$, whose endomorphism ring is $R = \mathbb{Q}$. Here, the module $R_R = \mathbb{Q}_{\mathbb{Q}}$ obviously has all of the above properties. On the other hand, $A_k = \mathbb{Q}_{\mathbb{Z}}$ has none of the properties, except injectivity. From (8.2), we see, therefore, that none of these properties, except perhaps injectivity, is ER. But even injectivity turns out to be non-ER, as the example $A = \mathbb{Z}/2\,\mathbb{Z}$ over $\mathbb{Z}$ shows. (In fact, we can take for A_k any non-injective simple module: Schur's Lemma implies that $R = \mathrm{End}(A_k)$ is a division ring, which is, of course, right injective over itself.) Finally, we leave it as an exercise for the reader to show that quasi-injectivity, as defined in §7, is also not an ER-property.

In doing mathematics, however, one should never give up too easily! If one looks only a little further, one will quickly find interesting examples of module properties that do turn out to be ER. After all, if we define a property $\mathcal{P}$ on A *by using* $R = \mathrm{End}(A_k)$ *alone with no reference to* A, then $\mathcal{P}$ is automatically an ER-property. For instance,

$\mathcal{P}$ could be "R is finite", "R is commutative", "R is a division ring", or "R is a local ring", etc. (Note that the last condition defines the ER-property "A_k is strongly indecomposable".) Examples not directly of the above nature are given below, where R will always denote $\mathrm{End}(A_k)$.

Examples 8.4. As a starter, the property "$A_k \neq 0$" is ER, since obviously it can be expressed by "$R \neq 0$". The "indecomposability" of the module A_k is also an ER-property, since it amounts to R being nonzero and having no nontrivial idempotents. Going a little bit beyond, we may consider the following module-theoretic properties.

(A) The property that A_k has a decomposition into a finite direct sum of indecomposable submodules is ER, since it can be expressed by saying that, in its endomorphism ring R, we can write $1 = e_1 + \cdots + e_n$ (for some n), where $e_1, \ldots, e_n$ are mutually orthogonal primitive idempotents.

(B) The property of A_k "having a finite Azumaya decomposition" (as defined in (5.2)) is also ER. In fact, it is known that A_k has this property iff R is a semiperfect ring in the sense of Bass; that is, R is semilocal, and idempotents of $R/\mathrm{rad}(R)$ can be lifted to idempotents in R (see [La$_1$: (23.8)]). Since this is a property of the ring R alone, the finite Azumaya decomposition property in question is ER.

(C) By definition, A_k is a *Fitting module* if Fitting's Lemma holds for A_k; that is, for any endomorphism $f \in R$, $A = \ker(f^n) \oplus \mathrm{im}(f^n)$ for some positive integer n. Armendariz, Fisher, and Snider proved in [AFS] that this holds iff the endomorphism ring R is strongly π-regular (that is, the sequence $aR \supseteq a^2R \supseteq a^3R \supseteq \cdots$ stabilizes for every $a \in R$). Thus, the property of being a Fitting module is also ER.

The examples in (8.4) above seem to suggest that, in considering direct sum decompositions of modules, the intervening module-theoretic properties tend to have a good chance of being ER. This brings us to the four key properties in the display (4.2) (substitution, cancellation, etc.), as well as the exchange property in §6, and the hopfian and cohopfian properties in §8. *Which of these properties are* ER? We shall try to provide answers to *almost all* cases of this question below. To begin with, we have the following.

Proposition 8.5. *"Dedekind-finite", "internal cancellation", and "substitution" are* ER*-properties.*

Proof. This is pretty easy for "Dedekind-finite". In fact, it is well-known (and rather straightforward to verify) that a module A_k is Dedekind-finite iff its endomorphism ring R is Dedekind-finite ($ab = 1 \Rightarrow ba = 1$ in R). The presence of such a characterization (purely in terms of R) makes it clear that "Dedekind-finite" (as a module property) is ER. Similarly, thanks to the result (4.4), the substitution property is also ER.

For "internal cancellation", we have to work a little harder. Given any module A_k, let $R = \mathrm{End}(A_k)$ as usual. Our goal is to show that A_k has internal cancellation iff R_R has internal cancellation; that is, iff the ring R has the following property:

(8.6) *For any two idempotents* $e,\, e' \in R$, $eR \cong e'R \Longrightarrow (1-e)R \cong (1-e')R.$

In the standard theory of idempotents, two idempotents $e,\, e' \in R$ are said to be *isomorphic* (written $e \cong e'$) if $eR \cong e'R$ (as right R-modules). It is known that this isomorphism notion for idempotents is left-right symmetric; in fact, we have $e \cong e'$ *iff there exist two elements* $a,\, b \in R$ *such that* $e = ab$ *and* $e' = ba$. (For a detailed proof of this fact, see §21 in [La$_1$].) To show what we want, first assume that the ring $R = \mathrm{End}(A_k)$ has the property (8.6), and consider two decompositions $N \oplus K \cong N' \oplus K'$ of A_k, with $N \cong N'$. Let $e \in R$ be the projection of A onto N with respect to the first decomposition, and define $e' \in R$ similarly with respect to the second decomposition. From $N \cong N'$, one can use the above description of idempotent isomorphisms to show that $e \cong e'$ in R (and conversely also: see [La$_1$: p. 316]). By the property (8.6), we then have $1 - e \cong 1 - e'$, and this implies that $K \cong K'$ (as k-modules). This shows that A_k has internal cancellation, and the reverse implication (internal cancellation on A_k implies (8.6)) is done by simply reversing the argument above. □

We should like to mention that there is another, perhaps more natural way to check that internal cancellation is an ER property. Indeed, by a more precise reformulation of the proof of Theorem 2.3, one can show that *a module* A_k *is internally cancellable iff, in the ring* $R = \mathrm{End}(A_k)$, *every regular element is unit-regular.* This characterization of internal cancellation is explicitly stated by Guralnick and Lanski [GL], who also gave another characterization in terms of the notion of "pseudosimilarity" in the endomorphism ring R. Each of these alternative characterizations would, of course, yield a proof for the fact that internal cancellation on modules is an ER-property. Endomorphism rings of internally cancellable modules constitute a very interesting class of rings called IC rings ("rings with internal cancellation"), or "partially unit-regular rings". For a self-contained investigation of this class of rings and for more relevant bibliographical information, see [KL].

Remark 8.7. The method of relating (8.6) to internal cancellation in modules leads easily to the verification of other similar examples of ER-properties. For instance, as Facchini and Herbera have observed in [FH], the property that A_k has (up to isomorphisms) only finitely many direct summands, and the property that A_k has (up to isomorphisms) only finitely many direct sum decompositions into nonzero summands, are both ER. The latter property (which is the stronger of the two) is called the *almost Krull-Schmidt property* in [FH].

Let us now consider the one property in (4.2) that was not covered by (8.5), namely, the cancellation property. This case turns out to present a small surprise: I thank A. Facchini and D. Arnold for pointing out the following result.

Theorem 8.8. *The cancellation property on modules is not* ER.

Proof. We work over the ring $k = \mathbb{Z}$, and use the $\mathbb{Z}$-module A constructed in Example (3.2)(5). To be more specific, let A be the subgroup of $\mathbb{Q}$ generated by $1/p$, where p ranges over, say, the (infinite) set of primes $\equiv 3 \pmod 4$. According to (3.2)(5), A is *not* cancellable. To compute $R := \mathrm{End}(A)$, note that any $\varphi \in R$ is the restriction of an endomorphism of $\mathbb{Q}_{\mathbb{Z}}$ (since $\mathbb{Q}$ is injective over $\mathbb{Z}$), so φ is given by multiplication by a rational number r. But in order that $r\,A \subseteq A$, r must clearly be an integer. Thus, $R \cong \mathbb{Z}$, and according to (3.6), $R_R = \mathbb{Z}_{\mathbb{Z}}$ is cancellable. This shows that the cancellation property on modules is *not* ER. $\square$

Concerning the cancellability of subgroups of $\mathbb{Q}$ and the stable range of their endomorphism rings, Facchini pointed out to us the relevance of a theorem of Fuchs and Loonstra [FL]. According to this theorem (which is also stated and proved in [Arn: Th. (8.12)]), *a rank* 1 *torsionfree abelian group A is cancellable iff either $A \cong \mathbb{Z}$, or* $\mathrm{End}(A)$ *has stable range* 1 (that is, A has substitution, by (4.4)). Thus, as long as a subgroup $A \subseteq \mathbb{Q}$ is noncyclic and has endomorphism ring $R \cong \mathbb{Z}$, the Fuchs-Loonstra theorem guarantees that A is *not* cancellable (since R fails to have stable range 1 by (1.6)(2)). In the constructive proof of (8.8) above, what we did was precisely to have chosen such a group $A \subseteq \mathbb{Q}$ for which we can directly verify the failure of cancellation (by the method of Jónsson in (3.2)(5)).

Next we come to the exchange properties defined in §6. For the *finite* exchange property, we quote the following important result of Warfield.

Theorm 8.9. [Wa$_3$] *A module A_k has the finite exchange property iff R_R has the finite exchange property, iff $_RR$ has the finite exchange property, where* $R = \mathrm{End}(A_k)$.

A sketch of a quick proof of this result using an appropriate dosage of category theory can be found on p. 25 of [Ar$_3$]. Note that, in view of (8.2), the first "iff" statement in (8.9) is tantamount to the fact that *"finite exchange", as a module-theoretic property, is* ER. In fact, it was precisely results such as Warfield's theorem (8.9) that had first led us to the formulation of the notion of the (so-called) ER properties.

As to whether the *general* exchange property is an ER-property, the situation turns out to be nebulous, as the following proposition shows.

Proposition 8.10. *The exchange property on right modules is an* ER*-property iff "exchange" is equivalent to "finite exchange" on all right modules.*

Proof. For any right k-module A, let $R = \mathrm{End}(A_k)$. For the exchange property to be ER, the criterion is that A_k has exchange iff R_R has exchange. Since R_R is f.g. (in fact it is cyclic), the latter is equivalent to R_R having finite exchange (see Footnote 8), and by (8.9), this is equivalent to A_k having finite exchange. Thus, the condition for the exchange property to be ER is precisely that A_k has exchange iff A_k has finite exchange, for right modules A over any ring. $\square$

Since it is not known (see (6.8)(1)) if finite exchange modules are necessarily exchange modules, (8.10) means that *the question whether the exchange property is* ER *remains open.* For some bibliographical information on this open question, see item (E) in §9 below.

Warfield [Wa_3] defined a ring R to be an *exchange ring* if R_R satisfies (finite) exchange. By (8.9), this is a left-right symmetric notion, and for a module A over any ring, A has the finite exchange property iff $\mathrm{End}(A)$ is an exchange ring. Later, Nicholson [Ni] and Goodearl [GW: p. 167] showed that exchange rings are precisely the rings R with the following interesting idempotent property:

$$(\dagger) \qquad \forall r \in R,\ \exists\ e = e^2 \in R \textit{ such that } e \in rR \textit{ and } 1 - e \in (1-r)R.$$

In fact, Nicholson showed, independently of (8.9), that a module A has finite exchange iff its endomorphism ring R satisfies (†), iff R satisfies the left analogue of (†). This provides a second proof for the fact that finite exchange on modules is an ER-property, and it, in turn, *implies* Warfield's theorem (8.9).

Having stated Nicholson's results above, we can now return to Yu's Theorem (6.11), and give a sketch for the missing proof of the key implication (1) $\Rightarrow$ (3). This means proving that, for any finite exchange module A_k, internal cancellation implies substitution. Since all three properties are ER, it suffices to do the same for R_R, where $R = \mathrm{End}(A_k)$ is an exchange ring. According to (4.4), this boils down to proving that *an exchange ring R with internal cancellation for right ideals has stable range* 1. To check this, start any equation $aR + bR = R$; say $ax + by = 1$ (where $x, y \in R$). Applying (†) to the element $r := by \in R$, we get an idempotent $e \in byR$ such that $1 - e \in (1 - by)R = axR$. Write $e = bys$ and $1 - e = axt$, for suitable elements $s, t \in R$. From $a(xt) + e = 1$, one can then invoke the internal cancellation property on R_R to find an element $z \in R$ such that $a + ez \in \mathrm{U}(R)$. Since $a + ez = a + b(ysz)$, this gives what we want. For more details on the last step of this argument, see [Yu: p. 108]; some further refinements of this circle of ideas are given in [KL].

We shall now conclude this section with a quick discussion of the hopfian and cohopfian properties defined in §8. The consideration of these two module properties brings out the idea that the notion of ER-property can be split into two parts, as follows. Keeping in mind the interpretation $(*)$ of the ER property in (8.2)(3), let us say that a module property $\mathcal{P}$ is ER_1 if, for any module A_k (over any ring k) with endomorphism ring $R := \mathrm{End}(A_k)$, A_k satisfies $\mathcal{P}$ implies that R_R satisfies $\mathcal{P}$, and we'll say that $\mathcal{P}$ is ER_2 if the converse implication holds. Of course, $\mathcal{P}$ is an ER property iff it is both ER_1 and ER_2. Using this terminology, the situation for the hopfian and cohopfian properties can be summarized as follows.

Proposition 8.11. (1) *The hopfian property is* ER_1 *but not* ER_2. (2) *The cohopfian property is* ER_2 *but not* ER_1. *In particular, neither property is* ER.

Proof. As usual, we let $R = \mathrm{End}(A_k)$, where A_k is a right module over any ring k.

(1) Assume A_k is hopfian. Obviously, A_k is Dedekind-finite. By (8.5), R_R is also Dedekind-finite, and hence hopfian.[13] This checks that the hopfian property is ER_1. Now consider the $\mathbb{Z}$-module A given by the Prüfer p-group (for any prime p). As we

[13]It is easy to see that a *projective* module is Dedekind-finite iff it is hopfian.

have observed in (7.12), this module is not hopfian. However, its endomorphism ring is the ring of the p-adic integers, which (as a commutative ring) is Dedekind-finite, and therefore hopfian over itself (see Footnote 13). This shows that the hopfian property is *not* ER_2.

(2) Assume now R_R is cohopfian. Since $\mathrm{End}(R_R)$ consists of left multiplications by elements of R, this means that any non left-0-divisor in R is a unit. Let $f \in R$ be any injective endomorphism of A_k. For any $g \in R$ such that $fg = 0$, we have $0 = fg(a) = f(g(a)) \Rightarrow g(a) = 0$ for any $a \in A$, so $g = 0$. This means that f is not a left 0-divisor in R, so $f \in \mathrm{U}(R)$; that is, f is an automorphism of A. This shows that A_k is cohopfian, so we have checked that the cohopfian property is ER_2. Taking A to be the Prüfer p-group again (over $k = \mathbb{Z}$), we know (cf. (7.12)) that A_k is cohopfian. However, as we have observed in (1), $R = \mathrm{End}(A)$ is the ring of the p-adic integers, and multiplication by p is an injective R-endomorphism of R that is not surjective, so R_R is not cohopfian. This shows that the cohopfian property is *not* ER_1. $\square$

§9. Epilogue

Needless to say, in an elementary exposition like this, many important issues and results in the area under investigation have been completely left out. To make up partly for this, we shall mention below a number of other significant results about cancellation, exchange, and related matters that we have not had space to cover. It is our hope that this will give the reader some useful tips on further reading in this area. (Besides, which author would pass up the golden chance to rave about dozens of nice theorems without having to prove *any* of them?)

(A) Menal [Me] showed that, over a von Neumann regular ring k whose primitive factor rings are artinian, $\mathrm{sr}(\mathrm{End}_k(A)) = 1$ for any f.g. k-module A; equivalently, by (4.4), A has the substitution and cancellation properties.

(B) Camps and Dicks [CD] have shown that artinian modules have semilocal endomorphism rings. In particular, by (2.10) and (4.4), such modules have the substitution and cancellation properties. In [HS], Herbera and Shamsuddin have further generalized this result, showing that if a module A has finite uniform dimension and is cohopfian, or has finite co-uniform dimension and is hopfian, then $\mathrm{End}(A)$ is semilocal, and hence A has the substitution and cancellation properties. (The former case covers the Camps-Dicks result, and the latter case implies that noetherian modules with finite co-uniform dimension have the substitution property.[14]) Many other cases of modules with semilocal endomorphism rings are discussed in [Fa_2: Ch. 4].

(C) Facchini [Fa_1] has shown that uniserial modules (modules whose submodules form a chain) have semilocal endomorphism rings, so again they have the substitution and cancellation properties.

[14] The assumption on finite co-uniform dimension in the latter cannot be dropped, as the example $\mathbb{Z}$ (as a $\mathbb{Z}$-module) shows.

(D) In [Br], Brookfield studied the direct sum cancellation problem for a noetherian module A. The main result of [Br] states that, if $A \oplus B \cong A \oplus C$ (where A is noetherian), then B and C have isomorphic submodule series, in the sense that there exist submodule series

$$(0) = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_n = B \quad \text{and}$$
$$(0) = C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq C_n = C\,,$$

and a permutation π such that $B_i/B_{i-1} \cong C_{\pi(i)}/C_{\pi(i)-1}$ for all $i \in [1, n]$.

(E) To a considerable extent, the initial study of exchange modules was motivated by the problem of finding isomorphic refinements for different direct sum decompositions; e.g. see [Wa$_2$]. From the 70s on, exchange (and finite exchange) modules gained importance in their own right, and became a promising independent topic for research. In this article, we are interested in such modules mainly insofar as they impact the themes of cancellation, internal cancellation, and substitution, so of course we have hardly done justice to the theory of exchange modules itself. Many interesting classes of exchange modules have since been discovered; see, for instance, [ZZ], [Sto], [MM$_1$], [MM$_2$], and the work of Harada, Oshiro, Rizvi, Yamagata, and many others. Broadly speaking, the finite exchange property is more relevant to the main themes of this paper, while the study of the full exchange property leads naturally to representation-theoretic problems. *The thorny question whether finite exchange implies full exchange has remained open in general.* However, this has been proved for various classes of modules, including, for instance, modules that are direct sums of indecomposables ([ZZ]), and quasi-continuous modules ([OR], [MM$_3$]). A positive answer was also recently given, in [DT], for modules with abelian endomorphism rings. There seems to be an error in this paper, but the result has been reaffirmed by a different, and much shorter, proof by P. P. Nielsen [Nie].

(F) In §8, following Warfield [Wa$_3$], we have defined a ring R to be an *exchange ring* if R_R (or equivalently ${}_RR$) satisfies (finite) exchange. The class of exchange rings is quite broad. For instance, it includes (by (7.8)) all 1-sided self-injective rings, and according to Nicholson [Ni], it also includes the family of *semi-π-regular rings*; i.e. rings R such that (1) idempotents in $\overline{R} = R/\mathrm{rad}(R)$ can be lifted to idempotents in R, and (2) $\overline{R}$ is π-regular, in the sense that every element in $\overline{R}$ has a power that is (von Neumann) regular. In particular, π-regular rings themselves (including von Neumann regular rings) and Bass's semiperfect rings are examples of exchange rings. Right semiartinian rings are also exchange rings according to Baccella [B]. An important connection to the theory of operator algebras is provided by the fact that *a unital C^*-algebra is an exchange ring iff it has real rank* 0; see [AG: Th. 7.2]. More recently, Ara [Ar$_4$] has shown that, if R is a *purely infinite simple ring* (that is, R is not a division ring, and every nonzero $r \in R$ satisfies an equation $xry = 1$ for some $x, y \in R$), then R is also an exchange ring. Yet another interesting family of exchange rings is given by the *clean rings* (introduced and so christened by Nicholson in [Ni]): these are rings in which every element is the sum of a unit and an idempotent. For instance, unit-regular rings, strongly π-regular rings (defined in (H) below), and abelian exchange rings are clean rings, and according to [CY_1], semiperfect rings are precisely the clean (or exchange) rings with only *finite* sets of orthogonal idempotents.

(G) Exchange rings whose primitive factors are artinian, or whose nilpotent elements are of bounded nilpotency index, are shown to have stable range 1 in [Yu]. In view of results such as (5.5), it is of interest to characterize the exchange rings with stable range 1. In generalization of (5.5) ((1) $\Leftrightarrow$ (2)), Camillo and Yu [CY_2] have shown that, for any exchange ring R, $\mathrm{sr}(R) = 1$ iff every regular element of R is unit-regular, and H. Chen [Ch_1] showed that $\mathrm{sr}(R) = 1$ is also equivalent to $aR = bR \Rightarrow a \in b \cdot \mathrm{U}(R)$ (for $a, b \in R$). For other characterizations, see [Yu], [Ch_1, Ch_2, Ch_4], and [KL]. More recently, the case of stable range n has been studied over exchange rings; see, for instance, [Ch_1], [CL_1, CL_2], [CC], and [WX]; see also items (J) and (K) below.

(H) We have shown in (2.12) that 0-dimensional commutative rings have stable range 1. This result has a wonderful (though much deeper) noncommutative generalization. A ring R is said to be *strongly π-regular* if, for any $a \in R$, we have $a^n \in a^{n+1}R$ for some n. (This condition is known to be left/right symmetric by a result of Dischinger.) If R is commutative, this condition is equivalent to R having dimension 0 [La_2: Ex. (4.15)]. In [Ar_2], Ara has shown that *strongly π-regular rings have stable range* 1, in generalization of (2.12). Coupling this with a result of Armendariz, Fisher, and Snider [AFS], it follows that any Fitting module has the substitution and cancellation properties.[15] From Ara's result and (5.5) above, we also see that a strongly π-regular von Neumann regular ring must be unit-regular, retrieving a result of Goodearl and Menal from [GM].

(I) Warfield [Wa_4] has generalized the basic connection between the substitution property and stable range 1 endomorphism rings to the case of stable range n. The upshot of this is that modules with endomorphism rings of stable range n are characterized by a certain "n-substitution property". Using this generalization, Warfield obtained the following cancellation theorem: *if* $\mathrm{End}(A)$ *has stable range* n *and* $A \oplus B \cong A \oplus C$, *then* $B \cong C$ *provided that* B *has a direct summand isomorphic to* $n \cdot A$. In the case of stable range 1, Warfield's results retrieve (4.4) and (5.1), although this may not be apparent from the above statement of Warfield's cancellation theorem. Some "weak" cancellation problems related to those of the above type were studied over exchange rings in [WX].

(J) The determination of the possible (exact) values for the stable range of a von Neumann regular ring seems difficult. For certain classes of von Neumann regular rings R, Menal and Moncasi [M] and Ara [Ar_1] have shown that, if $\mathrm{sr}(R) < \infty$, then $\mathrm{sr}(R) \le 2$ (and that indeed $\mathrm{sr}(R) = 2$ is possible); see also [Ch_3]. However, no von Neumann regular rings are known that have a *finite* stable range > 2. This leads to the open question: *does there exist a von Neumann regular ring (or at least an exchange ring) of stable range n for each integer $n \ge 3$?*

[15] In fact, taking for granted the Warfield-Fuchs result in (4.4) and the Armendariz-Fisher-Snider result in (8.4)(C), the theorem of Ara is tantamount to the fact that *Fitting modules satisfy substitution.* In this manner, a purely ring-theoretic theorem is turned into a purely module-theoretic one!

(K) To unify the study of various kinds of cancellation problems over exchange rings, Ara, Goodearl, O'Meara, and Pardo have introduced the notion of "separative cancellation": they called a ring R *separative* if, for all f.g. projective right R-modules A, B,

$$A \oplus A \cong A \oplus B \cong B \oplus B \Longrightarrow A \cong B. \tag{9.1}$$

(Note that this is a considerably weaker condition than A or B being cancellable.) In [AG], Ara et al. showed that, for exchange rings, *separativity is preserved by ring extensions*; this provides a very powerful method for the construction of separative exchange rings. Furthermore, it is shown that, *if an exchange ring R is separative, then* $\mathrm{sr}(R)$ *can only take the values* 1, 2, or ∞. (If R is, in addition, a simple ring, Chen and Li [CL_1] showed that $\mathrm{sr}(R)$ must be either 1 or ∞.) Since it is still conceivable that exchange rings are separative (no examples being known to the contrary), this last result *might* apply to all exchange rings. In that case, the results in [AG] would not only solve the stable range problem for von Neumann regular rings posed at the end of (J) above, but also settle a number of other open problems on regular rings.

(L) In commutative algebra, a time-honored research topic is the cancellation problem in the category of f.g. projective modules over affine algebras. As we have mentioned in the Introduction, the earliest manifestation of this problem was the famous "Serre's Conjecture", to the effect that f.g. projective modules are free over $R = k[x_1, \ldots, x_n]$, where k is any field (or a commutative PID).[16] This was proved independently by Quillen and Suslin in 1976. The work of these authors has been continued by many others, who study different aspects of the cancellation of projective modules over affine algebras, with very rich and interesting results. In general, however, this research (and other related work on the cancellation problem in the commutative case) involves methods and techniques rather different from those reported here. For a survey of this area of work, see the last chapter in the author's forthcoming book [La_6].

The above is only a very small sampling of the many beautiful results obtained on stable range, cancellation, and exchange that we have not managed to present. We hope this gives a flavor of the considerable amount of research that has been done on cancellation problems and related themes. There are numerous other surveys and expositions available; we would like to sign off by mentioning some of them. Wiegand's article [Wi] contains a survey of various important cancellation results (mostly) in the commutative setting, while Arnold's book [Arn] treats in detail many basic issues related to direct sum decompositions and cancellation in the category of finite-rank torsionfree abelian groups. (See especially §8 in [Arn], which is devoted to the discussion of the substitution and exchange properties of abelian groups, and the stable range of their endomorphism rings.) Goodearl's book [Go_3] provides an authoritative exposition on many cancellation (and n-cancellation) results over von Neumann regular rings. (The up-date section "In the Meantime ..." for the second edition offers an especially thorough guide to the progress on open problems on von Neumann regular rings before 1991.) A brief survey

[16] This amounts to the cancellability of R_R among f.g. projective modules over the polynomial ring R.

on stable range and its applications, covering some of the topics discussed above, can be found in [La_5], while a self-contained full length exposition on the basic theory is available in [CM]. For a more detailed survey on stability properties, especially of exchange rings and from a K-theoretic perspective, see [Ar_3]. For surveys and expositions on the theory of exchange modules, see [Har], [MM_2], and [Tu]. Facchini's book [Fa_2] is an excellent (and very complete) reference on all matters related to endomorphism rings, serial rings, and the Krull-Schmidt decompositions of modules. Another recent source containing much valuable information on module theory in general through concrete examples is Ringel's article [Ri]. Described as "polemical and exciting" by its reviewer (see MR 2002d: 16002), this paper provides highly educational and very inspiring reading to anyone interested in representation theory involving modules possibly of infinite length.

Acknowledgment. My hearty thanks are due to Professors P. Ara, D. M. Arnold, N. V. Dung, A. Facchini, K. R. Goodearl and D. Herbera, for their valuable comments, corrections, and suggestions on this article. Their collective feedback has led to very substantial improvements on the quality and accuracy of this work.

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