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A Prime Ideal Principle in commutative algebra

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Abstract

In this paper, we offer a general Prime Ideal Principle for proving that certain ideals in a commutative ring are prime. This leads to a direct and uniform treatment of a number of standard results on prime ideals in commutative algebra, due to Krull, Cohen, Kaplansky, Herstein, Isaacs, McAdam, D.D. Anderson, and others. More significantly, the simple nature of this Prime Ideal Principle enables us to generate a large number of hitherto unknown results of the "maximal implies prime" variety. The key notions used in our uniform approach to such prime ideal problems are those of *Oka families* and *Ako families* of ideals in a commutative ring, defined in (2.1) and (2.2). Much of this work has also natural interpretations in terms of categories of cyclic modules.

Keywords: Commutative algebra; Commutative rings; Prime ideals; Ideal families; Prime ideal principles; Module categories

1. Introduction

One of the most basic results in commutative algebra, given as the first theorem in Kaplansky's book [Ka_2], is (1.1) below, which guarantees that certain kinds of ideals in a commutative ring are prime. (In the following, all rings are assumed to be commutative with unity, unless otherwise specified.)

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Theorem 1.1. *(See [Ka$_2$, p. 1].) Let S be a multiplicative set in a ring R. An ideal $I \subseteq R$ that is maximal with respect to being disjoint from S is prime.*

Kaplansky credited this result to W. Krull. (We thank the referee for pointing us to the reference [Kr, Lemma, p. 732].) Shortly after presenting this result, Kaplansky stated: "In the next two theorems we exhibit two ways of constructing prime ideals without using a multiplicatively closed set." These theorems, due respectively to Cohen and Herstein, are as follows.

Theorem 1.2. *(See [Co].) Let I be an ideal in a ring R that is maximal with respect to not being finitely generated. Then I is prime.*

Theorem 1.3. *For an R-module M, let I be an ideal in R that is maximal among all annihilators of nonzero elements of M. Then I is prime.*

As an analogue to (1.2), Kaplansky also gave the following as Exercise 10 in [Ka$_2$, p. 8], which he attributed to I.M. Isaacs.

Theorem 1.4. *Let I be an ideal in a ring R that is maximal with respect to being nonprincipal. Then I is prime.*

The proofs of (1.1)–(1.4) given in the standard texts in commutative algebra (e.g. [Ei,Ka$_2$,Ma, Na$_1$]) were basically the same, but gave no indication as to whether (or better, how) these results may be related to one another. There were various other results too, all of the "maximal implies prime" variety, which are scattered in the literature (some of them having appeared, for instance, as exercises in Kaplansky's book [Ka$_2$]). But again, each one of these results seemed to have required a special twist for its proof; no clear unifying pattern inherent in this body of results has been proffered or discerned.

In this paper, we introduce an elementary Prime Ideal Principle, which states that, *for suitable ideal families $\mathcal{F}$ in a* (*commutative*) *ring, every ideal maximal with respect to not being in $\mathcal{F}$ is prime.* This Principle not only subsumes and unifies the results (1.1)–(1.4), but also applies readily to retrieve *all* other results of the same kind in the literature that the authors are aware of. More significantly, the simple nature of this Prime Ideal Principle enables us to generate with minimal effort a number of hitherto unknown results on the existence of prime (and maximal) ideals, with applications.

The key notion making this work possible is that of an *Oka family of ideals* in a ring, defined in (2.1) below. The idea of an Oka family can be traced back to a certain *Corollaire* in "Number VIII" in K. Oka's long series of papers on Cartan's theory of analytic functions in several complex variables, ca. 1951. Oka's *Corollaire* 2 [Ok, p. 209] was well hidden as a result on f.g. (finitely generated) ideals stated only for rings of complex functions in the SCV context. A clear statement of Oka's result in the general setting of commutative rings apparently first appeared in (3.3) of Nagata's book "Local Rings" [Na$_1$]. Using this result, Nagata gave a proof for Cohen's theorem [Co, Theorem 2] that a commutative ring is noetherian if its prime ideals are all f.g. [Na$_1$, (3.4)]. Our definition of an Oka family in (2.1) was directly inspired by Nagata's treatment. In (2.2), we also introduce the closely related notion of an *Ako family of ideals*—in a light-hearted reference to an Oka family. For both kinds of ideal families, the Prime Ideal Principle is stated and proved in (2.4), and a useful Supplement to this principle is given in (2.6). These, together with the fundamental result (2.7) explicating the logical dependence between the

Oka and Ako notions (and some of their stronger versions), constitute the theoretic backbone of this paper.

In Section 3, we give applications of the Prime Ideal Principle by first deriving uniformly all known cases of the "maximal implies prime" results that we are aware of. A rather pleasant fact here is that even D.D. Anderson's theorem on minimal primes in [An] turned out to be just a special case of the Prime Ideal Principle. Various new cases of applications of this Principle are then taken up in the second half of Section 3. For instance, by working with suitable new Oka families, we derive the following sufficient conditions for *maximal* ideals in a general commutative ring (see, respectively, (3.25), (3.24), and (3.22)):

(1) an ideal maximal with respect to not being a direct summand is maximal;
(2) an ideal maximal with respect to not being idempotent is maximal; and
(3) an ideal M maximal with respect to the property $M \supsetneq M^2 \supsetneq \cdots$ is maximal.

Let $\mathfrak{M}_c(R)$ denote the category of cyclic modules over a ring R. In Sections 4, 5, after setting up the correspondence between ideal families in R and subcategories of $\mathfrak{M}_c(R)$, we revisit the many types of ideal families introduced in Section 2, and give categorical interpretations for the defining properties of some of these families. Most notably, an Oka family of ideals in R is seen to correspond to a subcategory of $\mathfrak{M}_c(R)$ that is "closed under extensions." With this categorical view of Oka families, many examples of such families studied in Section 3 turn out to correspond to various familiar subcategories of $\mathfrak{M}_c(R)$ that are "clearly" closed under extensions from the module-theoretic viewpoint. For instance, the Oka family of f.g. ideals in R corresponds to the category of finitely presented cyclic modules, and the Oka family of direct summands in R corresponds to the category of projective cyclic modules, etc. On the other hand, *other* examples of subcategories of $\mathfrak{M}_c(R)$ that are already well known to be closed under extensions lead to further interesting examples of Oka families in R!

Throughout this paper, we use the notation $I \lhd R$ to indicate the fact that I is an ideal of a (commutative) ring R. For subsets $I, J, \ldots \subseteq R$, the ideal generated by their union is denoted by $(I, J, \ldots)$. For instance, if $a \in R$ and $I, J \lhd R$, we have $(I, J) = I + J$ and $(I, a) = I + (a)$. For $I \lhd R$ and $A \subseteq R$, we define $(I : A)$ to be the ideal $\{r \in R\colon rA \subseteq I\}$. The symbols $\mathrm{Spec}(R)$ and $\mathrm{Max}(R)$ shall denote, as usual, the prime ideal spectrum and the maximal ideal spectrum of the ring R.

Let $\mathcal{F}$ be a family of ideals in R with $R \in \mathcal{F}$. We say

(1) $\mathcal{F}$ is a *semifilter* if, for all $I, J \lhd R$, $I \supseteq J \in \mathcal{F} \Rightarrow I \in \mathcal{F}$;
(2) $\mathcal{F}$ is a *filter* if it is a semifilter and $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$; and
(3) $\mathcal{F}$ is *monoidal* if $A, B \in \mathcal{F} \Rightarrow AB \in \mathcal{F}$; that is, $\mathcal{F}$ is a submonoid of the monoid of all ideals of R under multiplication.

We will write $\mathcal{F}'$ for the complement of $\mathcal{F}$ (consisting of all ideals of R not belonging to $\mathcal{F}$), and $\mathrm{Max}(\mathcal{F}')$ for the set of maximal elements of $\mathcal{F}'$ (with respect to the partial ordering given by the inclusion of ideals). We will say $\mathcal{F}'$ is an MP-family ("maximal implies prime") if $\mathrm{Max}(\mathcal{F}') \subseteq \mathrm{Spec}(R)$. In this terminology, the Prime Ideal Principle simply states that, for any Oka or Ako family $\mathcal{F}$ (in any ring), $\mathcal{F}'$ is an MP-family.

2. Ideal families and the Prime Ideal Principle

We start with the two crucial definitions needed for this paper.

Definition 2.1. An ideal family $\mathcal{F}$ in a ring R with $R \in \mathcal{F}$ is said to be an *Oka family* (respectively *strongly Oka family*) if, for $a \in R$ and $I, A \lhd R$, (I, a), $(I : a) \in \mathcal{F} \Rightarrow I \in \mathcal{F}$ (respectively (I, A), $(I : A) \in \mathcal{F} \Rightarrow I \in \mathcal{F}$).

Definition 2.2. An ideal family $\mathcal{F}$ in a ring R with $R \in \mathcal{F}$ is said to be an *Ako family* (respectively *strongly Ako family*) if, for $a, b \in R$ and $I, B \lhd R$, (I, a), $(I, b) \in \mathcal{F} \Rightarrow (I, ab) \in \mathcal{F}$ (respectively (I, a), $(I, B) \in \mathcal{F} \Rightarrow (I, aB) \in \mathcal{F}$).

The following connections between the definitions given in (2.1)–(2.2) are self-evident.

Proposition 2.3.

(1) *A strongly Oka family is Oka, and conversely if R is a principal ideal ring.*
(2) *A strongly Ako family is Ako, and conversely if R is a principal ideal ring.*

With the notions of Oka and Ako families in place, we can now formulate the following general result.

Prime Ideal Principle 2.4. *If $\mathcal{F}$ is an Oka family or an Ako family, then $\mathcal{F}'$ is an MP-family*; *that is,* $\mathrm{Max}(\mathcal{F}') \subseteq \mathrm{Spec}(R)$.

Proof. Suppose some $I \in \mathrm{Max}(\mathcal{F}')$ is not prime. Since $I \neq R$, there exist $a, b \notin I$ such that $ab \in I$. Then $(I, b), (I : a) \supsetneq I$ (since they contain b), and $(I, a) \supsetneq I$ (since it contains a). Therefore, (I, a), (I, b) and $(I : a)$ all belong to $\mathcal{F}$. But $I = (I, ab) \notin \mathcal{F}$, so $\mathcal{F}$ is neither Oka nor Ako. $\square$

Remark 2.5. The converse of (2.4) does not hold in general. For instance, let $(R, (\pi))$ be a discrete valuation ring, and let $\mathcal{F} = \{(\pi)^i\colon\ i \neq 1, 4\}$. Then $\mathcal{F}' = \{(0), (\pi), (\pi)^4\}$ is an MP-family (since $\mathrm{Max}(\mathcal{F}') = \{\mathfrak{m}\} \subseteq \mathrm{Spec}(R)$). But $\mathcal{F}$ is *not* Oka since $I := (\pi)^4 \notin \mathcal{F}$, but $(I, \pi^2) = (I : \pi^2) = (\pi)^2 \in \mathcal{F}$. From $(I, \pi^4) = (\pi^4) \notin \mathcal{F}$, we see that $\mathcal{F}$ is also not Ako.

Another general statement concerning Oka and Ako families $\mathcal{F}$ is (2.6) below; this should be viewed as a supplement to (2.4). In a special case, it says that, under a suitable chain assumption on $\mathcal{F}'$, if the prime ideals of a ring "behave in certain ways," then all ideals "behave in the same way."

Prime Ideal Principle Supplement 2.6. *Let $\mathcal{F}$ be an Oka family or an Ako family in R. Assume that every nonempty chain of ideals in $\mathcal{F}'$* (*with respect to inclusion*) *has an upper bound in $\mathcal{F}'$.* (*This holds, for instance, if all ideals in $\mathcal{F}$ are f.g.*)

(1) *Let $\mathcal{F}_0$ be a semifilter of ideals in R. If every prime ideal in $\mathcal{F}_0$ belongs to $\mathcal{F}$, then $\mathcal{F}_0 \subseteq \mathcal{F}$.*
(2) *Let $J \lhd R$. If all prime ideals containing J* (*respectively properly containing J*) *are in $\mathcal{F}$, then all ideals containing J* (*respectively properly containing J*) *are in $\mathcal{F}$.*
(3) *If all prime ideals belong to $\mathcal{F}$, then all ideals belong to $\mathcal{F}$.*

Proof. (1) Assume there exists $I \in \mathcal{F}_0 \setminus \mathcal{F}$. By the hypothesis on $\mathcal{F}'$ (together with Zorn's Lemma), I is contained in some $P \in \operatorname{Max}(\mathcal{F}')$. Since $\mathcal{F}_0$ is a semifilter, $P \in \mathcal{F}_0$. But by (2.4), P is prime, and hence (by assumption) $P \in \mathcal{F}$, a contradiction.

(2) follows from (1) by taking $\mathcal{F}_0$ to be the semifilter consisting of all ideals containing J (respectively properly containing J). Finally, (3) follows from (1) by taking $\mathcal{F}_0$ to be the family of all ideals in R. $\square$

Before we proceed further, we would like to point out that, in our present approach, the Oka and Ako properties are being singled out as suitable "common denominators" for various other properties that would lead to a Prime Ideal Principle. The following result gives a sampling of some of these other stronger properties.

Logical Dependence Theorem 2.7. *For an ideal family $\mathcal{F}$ in R with $R \in \mathcal{F}$, consider the following properties, where A, B, I, J denote arbitrary ideals in R*:

(P_1) *$\mathcal{F}$ is a monoidal filter.*
(P_2) *$\mathcal{F}$ is monoidal, and $J \in \mathcal{F}$, $I \supseteq J \supseteq I^2 \Rightarrow I \in \mathcal{F}$.*
(P_3) $(I, A), (I, B) \in \mathcal{F} \Rightarrow (I, AB) \in \mathcal{F}$.
(Q_1) *$\mathcal{F}$ is a monoidal semifilter.*
(Q_2) *$\mathcal{F}$ is monoidal, and $J \in \mathcal{F}$, $I \supseteq J \supseteq I^n$ for some $n > 1 \Rightarrow I \in \mathcal{F}$.*
(Q_3) *$A, B \in \mathcal{F}$ and $AB \subseteq I \subseteq A \cap B \Rightarrow I \in \mathcal{F}$.*
(Q_4) *$A, B \in \mathcal{F}$ and $AB \subseteq I \subseteq A \cap B \Rightarrow I \in \mathcal{F}$ if A/I is cyclic.*
(Q_5) *$A, B \in \mathcal{F}$ and $AB \subseteq I \subseteq A \cap B \Rightarrow I \in \mathcal{F}$ if A/I, B/I are both cyclic.*
(O_4) *$I \subseteq J$ and $J, (I : J) \in \mathcal{F} \Rightarrow I \in \mathcal{F}$.*
(O_5) *$I \subseteq J$ and $J, (I : J) \in \mathcal{F} \Rightarrow I \in \mathcal{F}$ if J/I is cyclic.*[1]

We have the following chart of implications:

$$\begin{array}{ccccccccc}
(Q_1) & \Rightarrow & (Q_2) & \Rightarrow & (Q_3) & \Rightarrow & (Q_4) & \Rightarrow & (Q_5) \\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow \\
(P_1) & \Rightarrow & (P_2) & \Rightarrow & (P_3) & \Rightarrow & \text{str. Ako} & \Rightarrow & \text{Ako} \\
 & & & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & & & \text{str. Oka} & \Rightarrow & \text{Oka} & \Rightarrow & \text{P.I.P.} \\
 & & & & \Updownarrow & & \Updownarrow & & \\
 & & & & (O_4) & \Rightarrow & (O_5) & &
\end{array} \tag{2.8}$$

where P.I.P. *is short for "Prime Ideal Principle." Moreover, a family $\mathcal{F}$ satisfying* (P_3) (*with $R \in \mathcal{F}$*) *is closed under finite products and intersections.*

Proof. To begin with, it is easy to see that (P_1) $\Leftrightarrow$ (Q_1) $\Rightarrow$ (Q_2) $\Rightarrow$ (P_2). Now assume (P_2): we must prove (Q_2) and (P_3). For (P_3), let $(I, A), (I, B) \in \mathcal{F}$. The monoidal property gives $(I, A)(I, B) \in \mathcal{F}$, so

$$(I, AB) \supseteq (I, A)(I, B) \supseteq (I, AB)^2 \tag{2.9}$$

[1] To forestall misunderstanding, we remind the reader that all conditions (P_i) and (Q_i) are supposed to carry with them the presupposition that $R \in \mathcal{F}$.

implies that $(I, AB) \in \mathcal{F}$ according to (P_2). To prove (Q_2), suppose $I \supseteq J \supseteq I^n$ for some $n > 1$, where $J \in \mathcal{F}$ but $I \notin \mathcal{F}$. Then there exists a largest integer k such that $J + I^k \notin \mathcal{F}$ (noting that $J + I = I \notin \mathcal{F}$ and $J + I^n = J \in \mathcal{F}$). By (P_2), however,

$$J + I^k \supseteq J + I^{k+1} \supseteq \left(J + I^k\right)^2 \tag{2.9$'$}$$

together with $J + I^{k+1} \in \mathcal{F}$ imply that $J + I^k \in \mathcal{F}$, a contradiction.

Next, we prove $(P_3) \Leftrightarrow (Q_3)$. Assume (P_3), and let A, B, I be as in (Q_3). Then $(I, A) = A$ and $(I, B) = B$ are both in $\mathcal{F}$. By (P_3), we have $(I, AB) \in \mathcal{F}$. Since $AB \subseteq I$, this gives $I \in \mathcal{F}$. Conversely, assume (Q_3), and let I, A, B be as in (P_3). Let $A_0 = (I, A)$ and $B_0 = (I, B)$, which are both in $\mathcal{F}$. Noting that

$$A_0 B_0 = I^2 + AI + IB + AB \subseteq (I, AB) \subseteq A_0 \cap B_0, \tag{2.10}$$

we conclude from (Q_3) that $(I, AB) \in \mathcal{F}$.

Our notation scheme suggests that "strongly Ako" may be labeled (P_4), and "Ako" may be labeled (P_5), in which case we have trivially $(P_3) \Rightarrow (P_4) \Rightarrow (P_5)$. With such labellings, $(P_4) \Leftrightarrow (Q_4)$ and $(P_5) \Leftrightarrow (Q_5)$ may be proved in the same way as in the last paragraph.

The equivalence "strongly Oka $\Leftrightarrow$ (O_4)" is easily seen by writing $J = (I, A)$ in Definition 2.1 and noting that, with this notation, $(I : J) = (I : A)$. Specializing this to the case where J/I is cyclic, we get the equivalence "Oka $\Leftrightarrow$ (O_5)."

Finally, assume (P_3). Then its equivalent form (Q_3) implies that $\mathcal{F}$ is closed under finite products and intersections. To show that $\mathcal{F}$ is strongly Oka, assume (I, A) and $B = (I : A)$ both belong to $\mathcal{F}$. Since $(I, B) = B \in \mathcal{F}$, (P_3) implies that $(I, AB) \in \mathcal{F}$. But $AB \subseteq I$, so this amounts to $I \in \mathcal{F}$. Upon replacing A by a principal ideal (a), the same proof shows that a strongly Ako family is Oka. This (together with (2.4)) verifies all implications asserted in the chart (2.8). □

The observation in the result below gives a nice framework (as well as plenty of "abstract examples") for ideal families in any ring satisfying one of the properties in Theorem 2.7.

Proposition 2.11. *Let* (P) *be any of the properties* (P_j), (Q_j), *or* (O_j). *In any ring R, the class of ideal families having the property* (P) *is closed with respect to the formation of arbitrary intersections. In particular, any family of ideals in R "generates" a minimal ideal family that has the property* (P) *in R.*

Proof. The first conclusion is based on a routine check (for each of the properties), which we will leave to the reader. The second conclusion follows by taking the intersection of all ideal families containing the given ideals and satisfying the property (P). (Of course, it may well happen that the given ideals "generate" the family of *all* ideals.) □

As applications of (2.7), we will give below three easy constructions of strongly Oka and strongly Ako families in general rings—through the properties (P_1) and (P_3). Many more examples of Oka and Ako families will be given in the ensuing sections.

Proposition 2.12. *Let $\{\mathfrak{p}_i\}$ be a fixed subset of* $\operatorname{Spec}(R)$. *Then the family $\mathcal{F} = \{I \lhd R\colon I \nsubseteq \mathfrak{p}_i$ for every $i\}$ has the property* (P_1). *In particular, $\mathcal{F}$ is a strongly Oka and strongly Ako filter.*

Proof. $\mathcal{F}$ being clearly a semifilter, we need only check that $\mathcal{F}$ is monoidal. (Recall that $(Q_1) \Leftrightarrow (P_1)$.) Let $A, B \in \mathcal{F}$. If $AB \notin \mathcal{F}$, we would have $AB \subseteq \mathfrak{p}_i$ for some i. But then one of A, B is in $\mathfrak{p}_i$, a contradiction. $\square$

For the $\mathcal{F}$ above, $\mathrm{Max}(\mathcal{F}')$ consists of the maximal members of the family $\{\mathfrak{p}_i\}$.

Proposition 2.13. *Let $\{\mathfrak{m}_i\}$ be a fixed subset of* $\mathrm{Max}(R)$. *Then the family $\mathcal{F} = \{I \lhd R\colon I \notin \{\mathfrak{m}_i\}\}$ has the property* (P_3). *In particular, $\mathcal{F}$ is a strongly Oka and strongly Ako monoidal family.*

Proof. Since $R \in \mathcal{F}$, it suffices to check the property (Q_3) in (2.7). Let $I \lhd R$ be such that $AB \subseteq I \subseteq A \cap B$, where $A, B \in \mathcal{F}$. If $I = \mathfrak{m}_i$ for some i, clearly $A = B = R$. But then $I \supseteq AB = R$, a contradiction. $\square$

For the family $\mathcal{F}$ in (2.13), of course $\mathcal{F}' = \mathrm{Max}(\mathcal{F}') = \{\mathfrak{m}_i\}$. We note however that, if R is not a field, then $(0) \notin \{\mathfrak{m}_i\}$, and hence $(0) \in \mathcal{F}$. Thus, $\mathcal{F}$ is *not* a semifilter (if the set $\{\mathfrak{m}_i\}$ is nonempty).

Proposition 2.14. *Let $\mathcal{S}$, $\mathcal{T}$ be ideal families in R such that $\mathcal{S}$ is closed under multiplications and $\mathcal{T}$ is closed under finite intersections. Let*

$$\mathcal{F} = \{R\} \cup \{J \lhd R\colon\ S \subseteq J \subseteq T \textit{ for some } S \in \mathcal{S} \textit{ and } T \in \mathcal{T}\}. \tag{2.15}$$

Then $\mathcal{F}$ has the property (P_3). *In particular, $\mathcal{F}$ is a strongly Oka and strongly Ako monoidal family.*

Proof. Again, we will check the property (Q_3). Let $I \lhd R$ be such that $AB \subseteq I \subseteq A \cap B$, where $A, B \in \mathcal{F}$. If $A = R$, we have $I = B \in \mathcal{F}$. We may thus assume that $A, B \subsetneq R$. Then $S \subseteq A \subseteq T$ and $S' \subseteq B \subseteq T'$ for suitable $S, S' \in \mathcal{S}$ and $T, T' \in \mathcal{T}$. Now $SS' \in \mathcal{S}$ and $T \cap T' \in \mathcal{T}$, so $SS' \subseteq I \subseteq T \cap T' \Rightarrow I \in \mathcal{F}$. $\square$

Example 2.16. If the family $\mathcal{T}$ happens to contain R, $\mathcal{F}$ in (2.15) is simply the (monoidal) semifilter generated by $\mathcal{S}$. For more concrete examples of (2.15), let $\mathcal{T}$ be a singleton family $\{T\}$, where $T \lhd R$. With T fixed, we may take, for instance, $\mathcal{S} = \{0\}$, or $\mathcal{S} = \{T^n\colon n \geqslant 1\}$. In this way, we get two families $\mathcal{F}$, consisting of R together with all subideals of T, or R together with all subideals of T containing some power of T. Both of these families have the (P_3) property. It is an interesting exercise to confirm the Prime Ideal Principle by directly computing the set $\mathrm{Max}(\mathcal{F}')$ in each of these cases.

Given the chart of implications in (2.8), various questions concerning the further relationships between the Oka and Ako families (and their strong analogues) naturally arise. For instance, are "Oka" and "Ako" logically independent properties? Is "Oka" equivalent to "strongly Oka", and is "Ako" equivalent to "strongly Ako," etc.? Since the main focus of this paper is the study of the Prime Ideal Principle, to take up these questions here would take us too far afield. We will thus postpone the investigation of such questions to [LR]. In this sequel to the present paper, we will show (among other things) that the chart of implications (2.8) is *complete,* in the sense that no new implication arrows can be added to the chart—other than those obtainable by compositions.

3. Applications of the Prime Ideal Principle

We start this section with some preliminary applications of (2.4) and (2.6) to the case of ideal families having the strongest property (P_1) in (2.7). Several of the conclusions we draw here are familiar facts in commutative algebra. However, even though some of these conclusions (e.g. a subset of those in (3.1)–(3.5)) are already known, they have not been previously recognized as results coming from a common source. *Here, they are all derived simply and uniformly from* (2.4) *and* (2.6). This work is quite easy to follow since it boils down to just a routine matter of checking the Oka property of a suitable ideal family (which usually satisfies even stronger properties). For instance, the easy and natural derivation of (3.4) from (3.3) is a case in point. The second half of this section offers more applications, via the construction of new Oka and Ako families in commutative rings.

The very first case of the application of (2.4) is where $\mathcal{F} = \{R\}$, which, of course, satisfies (P_1). In this case, we get the standard conclusion that $\mathrm{Max}(R) \subseteq \mathrm{Spec}(R)$. A bit more generally, we can easily retrieve (1.1) as a special case of (2.4), as follows.

Proposition 3.1. *Let $S \subseteq R$ be a nonempty multiplicative set. The ideal family $\mathcal{F} = \{I \lhd R\colon I \cap S \neq \emptyset\}$ has the* (*strongest*) *property* (P_1) *in* (2.7). *In particular, $\mathcal{F}$ is strongly Oka and strongly Ako, and* $\mathrm{Max}(\mathcal{F}') \subseteq \mathrm{Spec}(R)$.

As a special case, we can take S to be the multiplicative set of all non 0-divisors in R. In this case, $\mathcal{F}$ is the family of the (so-called) *regular ideals* [Hu, p. 1]. In this case, (3.1) and (2.6)(2) (with $J = 0$) give the following familiar conclusions.

Corollary 3.2.

(1) *An ideal maximal with respect to containing only* 0*-divisors is prime*;
(2) *if $R \neq 0$ and all nonzero prime ideals in R are regular, then R is an integral domain.*

For a second application, we start with a fixed set of ideals $\{I_j\}$ in R, and let $\mathcal{F}$ be the family of ideals that contain a (finite) product of the I_j's; that is, $\mathcal{F}$ is the monoidal filter (family with the property (P_1)) generated by the I_j's. By (2.7), $\mathcal{F}$ is (strongly) Ako and Oka, so (2.4) and (2.6)(3) immediately give the following.

Proposition 3.3. *An ideal maximal with respect to not containing a product of the I_j's is prime. If the I_j's are f.g., and every prime ideal of R contains some I_j, then some product of the I_j's is the zero ideal.*

Corollary 3.4. *(See D.D. Anderson [An].) Let $\mathcal{S} = \{I_j\}$ be the set of minimal primes in R. If each I_j is f.g., then $|\mathcal{S}| < \infty$.*

Proof. Since every prime contains a minimal prime, (3.3) gives an equation $I_{j_1} \cdots I_{j_n} = 0$. Clearly, $\mathcal{S} \subseteq \{I_{j_1}, \ldots, I_{j_n}\}$. (This is, of course, essentially the proof in [An], adapted to the axiomatic setting of this paper.) □

Professor L. Avramov has kindly informed us that Anderson's result can *also* be deduced directly from Cohen's Theorem 1.2, by localizing R at the multiplicative set $R \setminus \bigcup_j I_j$. This proof and the one given above bear an interesting comparison. The reduction to Cohen's theorem is nice, but uses the Prime Ideal Principle in its strongest form (for Oka families, after localization). The proof given above uses, however, only a weak form of the Prime Ideal Principle—for (P_1) families, and without localization.

We note in passing that, in the torsion theory of rings, a *Gabriel topology* on a (in our case commutative) ring R is a filter of ideals $\mathcal{F}$ in R satisfying a certain axiom (T_4); see [St, (VI.5)]. These conditions imply that $\mathcal{F}$ is monoidal [St, (VI.5.3)], so $\mathcal{F}$ is again a (P_1) family. In particular, the Prime Ideal Principle (2.4) applies to any Gabriel topology $\mathcal{F}$. Indeed, the fact that $\mathrm{Max}(\mathcal{F}') \subseteq \mathrm{Spec}(R)$ in this particular case was explicitly stated in [St, (VI.6.14)(i)].

Next, we consider the *point annihilators* of an R-module M. By definition, a point annihilator of M is an ideal of the form $\mathrm{ann}(m)$, where $m \in M \setminus \{0\}$.

Proposition 3.5. *Let M be a fixed R-module, and let $S \subseteq R$ be a multiplicative set containing* 1 *but not* 0. *Let $\mathcal{F}$ be the family of ideals $I \lhd R$ such that, for $m \in M$, $I \cdot m = 0 \Rightarrow sm = 0$ for some $s \in S$. Then $\mathcal{F}$ is a strongly Ako semifilter; in particular, it is both Ako and Oka. If an ideal $J \lhd R$ is maximal with respect to being a point annihilator of M disjoint from S, then $J \in \mathrm{Spec}(R)$. In particular, taking $S = \{1\}$, maximal point annihilators of M are prime: this retrieves Herstein's result stated in* (1.3).

Proof. $\mathcal{F}$ is clearly a semifilter. To check that it is strongly Ako, let $a \in R$ and $I, B \lhd R$ be such that both (I, a) and (I, B) belong to $\mathcal{F}$. To show $(I, aB) \in \mathcal{F}$, suppose $(I, aB) \cdot m = 0$, where $m \in M$. Then $(I, B) \cdot am = 0$, so $sam = 0$ for some $s \in S$. From this, $(I, a) \cdot sm = 0$, so we have $s'sm = 0$ for some $s' \in S$. Since $s's \in S$, this shows that $(I, aB) \in \mathcal{F}$. The rest follows from the Prime Ideal Principle (2.4), and the (easy) fact that the ideals J in question are exactly those in $\mathrm{Max}(\mathcal{F}')$. $\square$

Remarks 3.6. (1) The primality of J was first proved by McAdam [Mc], although he stated it only in the case where S is the complement of a given prime ideal of R.

(2) As a supplement to (3.5), it is worth noting that, if either R is a noetherian ring or M is a noetherian module, then as in the previous examples, *the family $\mathcal{F}$ again has the property* (Q_1) (and hence (P_1)). First, $\mathcal{F}$ is clearly a semifilter. To show $\mathcal{F}$ is monoidal, let $A, B \in \mathcal{F}$, and say $BA \cdot m = 0$, where $m \in M$. If R is noetherian, $A = (a_1, \ldots, a_n)$ for some a_i's. Then $B \cdot a_i m = 0 \Rightarrow s_i a_i m = 0$ for suitable $s_i \in S$, and hence $sA \cdot m = 0$ for $s = s_1 \cdots s_n \in S$. This implies $s'sm = 0$ for some $s' \in S$, so we have checked that $BA \in \mathcal{F}$. If M is noetherian instead, a similar argument works. However, without assuming any finiteness conditions, one will have to use Proposition 3.5.

In the setting of (3.5), with M and S given, we can define the *S-torsion submodule* of M to be

$$M_t := \{m \in M\colon\ sm = 0 \text{ for some } s \in S\}. \tag{3.7}$$

Note that this is exactly the kernel of the localization map $M \to S^{-1}M$. With the above notation, the $\mathcal{F}$ in (3.5) is just the family of ideals $I \lhd R$ such that $\mathrm{ann}_M(I) \subseteq M_t$. Let us see what happens when we apply this setup to the R-module $M = R$. In this case, the ideals $I \in \mathcal{F}$ (characterized by the property that $\mathrm{ann}(I) \subseteq R_t$) are said to be the *$S$-dense ideals*. Proposition 3.5 then gives the following.

Corollary 3.8. *The family of S-dense ideals in a ring R has the property* (P_1) *in* (2.7). *In particular, an ideal in R maximal with respect to not being S-dense is prime.*

In the case $S = \{1\}$, the S-dense ideals are just called *dense*; these are the $I \lhd R$ that are faithful as R-modules. For instance, a regular ideal is always dense. The converse does not hold in general, but does hold over a noetherian ring R; see, e.g. [Ka_2, Theorem 82]. Thus, in case R is noetherian and $S = \{1\}$, (3.8) merely recaptures (3.2).

We can also apply the Prime Ideal Principle to the family of essential ideals. To this end, we need the following lemma.

Lemma 3.9. *Let I_1, I_2 be essential ideals in a reduced ring R. Then $I_1 I_2$ is also essential.*

Proof. Let $r \neq 0$. Then $rx \in I_1 \setminus \{0\}$ for some $x \in R$, and therefore $rxy \in I_2 \setminus \{0\}$ for some $y \in R$. Since R is reduced, we have $0 \neq r(xrxy) = (rx)(rxy) \in I_1 I_2$. This shows that $I_1 I_2$ is essential. □

Remark 3.10. The proof above works already for essential *right ideals* I_1, I_2 in a possibly noncommutative ring R (using, in addition, the results [La_1,La_3: Ex.'s 12.14, 12.17]). However, the conclusion of the lemma is false in general if R is not assumed to be reduced. For instance, let $(R, \mathfrak{m})$ be a (commutative) local ring with $0 = \mathfrak{m}^2 \subsetneq \mathfrak{m}$. Then $\mathfrak{m}$ is essential in R, but $\mathfrak{m}^2 = 0$ is not.

Proposition 3.11. *Let R be a reduced ring. Then the family $\mathcal{F}$ of essential ideals in R has the* (*strongest*) *property* (P_1) *in* (2.7). *In particular, an ideal in R maximal with respect to being inessential in R is prime. If R is, in addition, noetherian, nonzero, and every nonzero prime is essential in R, then R is an integral domain.*

Proof. The property (P_1) follows easily from (3.9). To prove the last statement, note that the assumptions there imply that all nonzero ideals in R are essential, by (2.6)(2) (with $J = 0$). So all nonzero ideals lie in $\mathcal{F}$, and thus $(0) \in \mathrm{Max}(\mathcal{F}') \subseteq \mathrm{Spec}(R)$ by (2.4). □

The family of invertible ideals in a ring R fits well into our general axiomatic scheme as well, although it no longer has the full property (P_1). To analyze this family, we first prove the following Factorization Theorem, the case (2) of which is crucial for working with invertible ideals. The cases (3) and (4) in the theorem will be important for the later applications of the Prime Ideal Principle in this section, and are grouped together here with case (2) since these results are of the same spirit.

Factorization Theorem 3.12. *Let $I \subseteq J$ be ideals of R. The factorization equation $I = J \cdot (I : J)$ holds under each of the following assumptions*:

(1) *$J = R$, or $J = I$, or $I = 0$*;
(2) *J is an invertible ideal*;
(3) *J is a principal ideal*;
(4) *J/I is cyclic, and the ideals J and $J \cdot (I : J)$ are idempotent.*

Proof. (1) If $J = R$, $(I : J) = I$. If $J = I$, $(I : J) = R$. In either case, $I = J \cdot (I : J)$ obviously holds. The case $I = 0$ follows from $J \cdot (I : J) \subseteq I$.

(2) Since $J^{-1}I \lhd R$ and $J^{-1}I \cdot J \subseteq I$, we have $J^{-1}I \subseteq (I : J)$. Since $J \cdot J^{-1} = R$, multiplying this by J yields $I \subseteq J \cdot (I : J) \subseteq I$.

(3) Say $J = (x)$, where $x \in R$. From $I \subseteq J$, it is easy to see that $I = x \cdot (I : x)$. Therefore, $I = J \cdot (I : J)$.

(4) Say $J = (I, a)$, where $a \in R$. Let $K := J \cdot (I : J) \subseteq I$. By assumption, $K = K^2 \subseteq I^2 \subseteq J \cdot (I : J) = K$. Thus, $I^2 = K$. Our goal is to show that $I \subseteq K$. Let $i_1 \in I$. Since $i_1 \in J = J^2$, we can write $i_1 = i_2 + ar$ where $i_2 \in I^2$ and $r \in R$. We have then $r \in (I : a) = (I : J)$, and so $ar \in J \cdot (I : J) = K$. Now $i_1 = i_2 + ar \in I^2 + K = K$, as desired. $\square$

Remark 3.13. Needless to say, the factorization equation $I = J \cdot (I : J)$ fails in general. For instance, if I is a f.g. regular prime ideal and J is an ideal strictly between I and R, then $J \cdot (I : J) \subsetneq I$. (This is because $(I : J) = I$. If $I = J \cdot (I : J)$, then $I = J \cdot I$, and the determinant trick in commutative algebra would have given $J = R$.)

We can now state the result on invertible ideals. The conclusion $\mathrm{Max}(\mathcal{F}') \subseteq \mathrm{Spec}(R)$ below, from [Ka$_2$, Exercise 36, p. 44], is due to McAdam, in the special case of domains. Here, using (3.12)(2), we work with arbitrary monoidal families of invertible ideals, and prove our conclusions more generally for all commutative rings.

Proposition 3.14. *Any monoidal family $\mathcal{F}$ of invertible ideals in a nonzero ring R is strongly Oka. Therefore, $\mathcal{F}'$ is an* MP*-family. In general, however, such families $\mathcal{F}$ need not be Ako.*

Proof. We check the property (O_4) in (2.7). Let $I \subseteq J$ be ideals such that J and $(I : J)$ both belong to $\mathcal{F}$. By (3.12)(2), $I = J \cdot (I : J) \in \mathcal{F}$.

To see that $\mathcal{F}$ need not be Ako in general, consider the integral domain $R = \mathbb{Z}[\theta]$, where $\theta^2 = 5$, and let $\mathcal{F}$ be the monoidal family of *all* invertible ideals in R. Let $a = 2$ and $I = (2(1 + \theta)) \lhd R$. Then $(I, a) = 2R$ is invertible. However, (I, a^2) is not, since

$$\big(I, a^2\big) = \big(4, 2(1 + \theta)\big) = 2 \cdot (2, 1 + \theta),$$

and $(2, 1 + \theta)$ is *not* an invertible ideal in R, according to [La$_2$, (2.19C)]. This showed that $\mathcal{F}$ is not Ako. Note that the ring R in this example is not a Dedekind domain, since $(1 + \theta)/2 \notin R$ is integral over R. (In a Dedekind domain R, $\mathcal{F}$ would have been the family of all nonzero ideals in R, which, of course, would have satisfied (P_1).) $\square$

Corollary 3.15. *If all nonzero primes in a nonzero ring R are invertible, then R is a Dedekind domain.*

Proof. Let $\mathcal{F}$ be the family of all invertible ideals in R. By (3.14), $\mathcal{F}$ is (strongly) Oka. Also, all ideals in $\mathcal{F}$ are f.g. Thus, by (2.6)(2), the hypothesis implies that all nonzero ideals of R are in $\mathcal{F}$. Since invertible ideals are regular, this in turn implies that R is a domain, and hence a Dedekind domain. $\square$

Remark. In the special case of integral domains, the above result was first proved by I.S. Cohen: see [Co, Theorem 7].

We now turn our attention to families of ideals that are defined via the number of elements needed to generate them. More generally, for any R-module M, let $\mu(M)$ denote the least cardinal μ such that M can be generated by μ elements.

Proposition 3.16. *For a fixed infinite cardinal α, let $\mathcal{F}_\alpha$ (respectively $\mathcal{F}_{<\alpha}$) be the family of ideals $I \lhd R$ such that $\mu(I) \leqslant \alpha$ (respectively $\mu(I) < \alpha$). Then $\mathcal{F}_\alpha$ (respectively $\mathcal{F}_{<\alpha}$) is a monoidal Oka family, and we have* $\mathrm{Max}(\mathcal{F}'_\alpha) \subseteq \mathrm{Spec}(R)$ (*respectively* $\mathrm{Max}(\mathcal{F}'_{<\alpha}) \subseteq \mathrm{Spec}(R)$).

Proof. We first treat the case $\mathcal{F}_\alpha$, which is monoidal since $\alpha \cdot \alpha = \alpha$. Let $I \lhd R$ and $a \in R$ be such that $\mu((I, a)) \leqslant \alpha$ and $\mu((I : a)) \leqslant \alpha$. Then $(I, a) = (I_0, a)$ for some ideal $I_0 \subseteq I$ with $\mu(I_0) \leqslant \alpha$. It is easy to check that $I = I_0 + a(I : a)$. Thus, $\mu(I) \leqslant \alpha + \alpha = \alpha$; that is, $I \in \mathcal{F}_\alpha$. This shows that $\mathcal{F}_\alpha$ is Oka, so (2.4) gives $\mathrm{Max}(\mathcal{F}'_\alpha) \subseteq \mathrm{Spec}(R)$. The case of $\mathcal{F}_{<\alpha}$ can be treated similarly. □

In the case where $\alpha = \aleph_0$, $\mathcal{F}_{<\aleph_0}$ is the family of f.g. (finitely generated) ideals in R. The conclusion that this family is Oka was due to Oka [Ok, Corollary 2]. The proof given here is a streamlined version of those given in [Na_1, p. 8] and [Ka_2, p. 5].[2] In this case, (2.6) gives Cohen's famous theorem that *R is noetherian iff all primes of R are f.g.* [Co, Theorem 2]. On the other hand, the conclusions in (3.16) for the family $\mathcal{F}_{\aleph_0}$ were noted in Exercise 11 in [Ka_2, p. 8]. Surprisingly, however, there seems to be no reference in the literature for the cases where α is a general infinite cardinal.

One more case that can be treated by similar methods is where $\alpha = 1$. Following (3.16), we will write $\mathcal{F}_1$ for the family of *principal* ideals in a (given) ring R.

Proposition 3.17. *Let $S \subseteq R$ be a multiplicatively closed set containing* 1 (*and possibly also* 0)*, and let $\mathcal{F}$ be the family of principal ideals (s) where $s \in S$. Then $\mathcal{F}$ is a monoidal strongly Oka family, so we have* $\mathrm{Max}(\mathcal{F}') \subseteq \mathrm{Spec}(R)$. *In particular, these conclusions apply to the family $\mathcal{F}_1$ of all principal ideals in R, and R is a principal ideal ring iff all prime ideals of R are principal. The conclusions also apply to the family of principal ideals generated by non* 0*-divisors of R.*

Proof. $\mathcal{F}$ is monoidal since $(x)(y) = (xy)$, and $x, y \in S \Rightarrow xy \in S$. Let $I \subseteq J$ be ideals such that J and $(I : J)$ belong to $\mathcal{F}$. Since J is principal, (3.12)(3) implies that $I = J \cdot (I : J) \in \mathcal{F}$. This checks the ($O_4$) property in (2.7), so $\mathcal{F}$ is strongly Oka. The rest of (3.17) follows from (2.4) and (2.6). □

Again, some historical notes on (3.17) are in order. The part of this proposition on principal ideal rings was an observation of Kaplansky; see [Ka_1, Footnote 8]. The primality of an ideal maximal with respect to not being principal was attributed by Kaplansky to I.M. Isaacs; see Exercise 10 in [Ka_2, p. 8]. The (strong) Oka property for $\mathcal{F}$ (proved above for any given multiplicative set S) is a common source for all of the above.

Remark 3.18. The families studied in (3.16) and (3.17) are Oka families, but in general they are *not* Ako families. We can see this as follows. Let R be a ring with two elements a, b such that

[2] For another approach to this result from the viewpoint of module categories, see the proof of (5.4)(1) below.

$I = (a) \cap (b)$ is not f.g. (A well known example is the subring

$$R = \mathbb{Z}\big[2x, 2x^2, 2x^3, \ldots\big] \subseteq \mathbb{Z}[x], \tag{3.19}$$

with $a = 2x$ and $b = 2x^2$: see [BJ, p. 58] or [Gi, p. 47].) Let $\mathcal{F}$ be the family of principal (respectively f.g.) ideals in R. By (3.17) (respectively (3.16)), $\mathcal{F}$ is Oka. Here, $(I, a) = (a)$ and $(I, b) = (b)$ are both in $\mathcal{F}$. However, $(I, ab) = I \notin \mathcal{F}$, so $\mathcal{F}$ is *not* Ako. In case $\mathcal{F}$ is the family of all principal ideals, it is in fact a strongly Oka family by (3.17). Thus, we have here a second example of a strongly Oka family that is not Ako (and therefore not having the property (P_3)).

The example above definitely demonstrated the "superiority" of the Oka property over the Ako property, as the latter property has been shown to be insufficient to handle the two families $\mathcal{F}_{<\aleph_0}$ and $\mathcal{F}_1$ considered in (3.16) and (3.17).

To complete this section, we will give some examples of Oka families arising from the consideration of the descending chains of powers of ideals in a ring R. The following is a general observation on the intersections of such ideal powers.

Proposition 3.20. *Let $\mathcal{F}_0$ be a monoidal filter in R (that is, $\mathcal{F}_0$ is an ideal family with the property* (P_1)). *Then the family $\mathcal{F} := \{C \lhd R\colon \bigcap C^n \in \mathcal{F}_0\}$ is also a monoidal filter. In particular, any ideal J maximal with respect to the property $\bigcap J^n \notin \mathcal{F}_0$ is prime.*

Proof. $\mathcal{F}$ is clearly a semifilter containing R, so we need only check that $B, C \in \mathcal{F} \Rightarrow BC \in \mathcal{F}$. This follows from the observation that

$$\bigcap (BC)^m = \bigcap \big(B^m C^m\big) \supseteq \Big(\bigcap B^m\Big)\Big(\bigcap C^m\Big) \in \mathcal{F}_0. \qquad \square \tag{3.21}$$

For instance, if S is any multiplicative set in R, (3.20) is applicable to $\mathcal{F}_0 = \{I \lhd R\colon I \cap S \neq \emptyset\}$. Thus, any ideal J maximal with respect to $S \cap (\bigcap J^n) = \emptyset$ is prime. In particular, if R is an integral domain, we can take S to be $R \setminus \{0\}$. In this case, we see that any ideal J maximal with respect to $\bigcap J^n = 0$ is prime.

Proposition 3.22. *The family $\mathcal{F}$ of ideals $C \lhd R$ such that the chain $C \supseteq C^2 \supseteq \cdots$ stabilizes has the property* (Q_2) *in* (2.7). *Any ideal M maximal with respect to the property $M \supsetneq M^2 \supsetneq \cdots$ is a maximal ideal. A noetherian ring R is artinian iff $C \supseteq C^2 \supseteq \cdots$ stabilizes for every $C \in \mathrm{Max}(R)$.*

Proof. Again, we have $R \in \mathcal{F}$. To check the monoidal property, let $B, C \in \mathcal{F}$. For some $m \geqslant 1$, we have $B^m = B^{m+1}$ and $C^m = C^{m+1}$. Thus,

$$(BC)^m = B^m C^m = B^{m+1} C^{m+1} = (BC)^{m+1}, \tag{3.23}$$

so $BC \in \mathcal{F}$. To check the second half of (Q_2), let $C \in \mathcal{F}$ and $I \supseteq C \supseteq I^n$, where $I \lhd R$ and $n > 1$. Again, say $C^m = C^{m+1}$. Then $C^m \supseteq (I^n)^m \supseteq C^{nm} = C^m$ gives $I^{nm} = C^m$. It follows that $(I^{nm})^2 = C^{2m} = C^m = I^{nm}$, so $I \in \mathcal{F}$. Thus, (2.4) applies, showing that the ideal M in (3.22) is prime. But in the domain $\overline{R} = R/M$, $(\bar{a}) \supseteq (\bar{a})^2 \supseteq \cdots$ stabilizes for every $(\bar{a})$. This means that $\overline{R}$ is a field, so M is maximal.

Finally, assume that R is noetherian with $\mathrm{Max}(R) \subseteq \mathcal{F}$. The last paragraph shows that $\mathcal{F}' = \emptyset$, so $\mathcal{F}$ is the family of *all* ideals. This readily implies that $\dim R = 0$, and hence R is artinian. (In any case, dim $R = 0$ could have been easily checked via localizations and Krull's Intersection Theorem.) $\square$

While we can think of the ideals $C \in \mathcal{F}$ above as those having a suitable (positive) power that is idempotent, we can equally well work with families of idempotent ideals themselves. In this case, we get the following result.

Proposition 3.24. *Let $\mathcal{F}$ be a monoidal family of idempotent ideals. Then $\mathcal{F}$ is an Oka family, and so $\mathcal{F}'$ is an* MP*-family. If an ideal $M \lhd R$ is maximal with respect to the property $M \neq M^2$, then $M \in \mathrm{Max}(R)$.*

Proof. It suffices to check that $\mathcal{F}$ has the property (O_5) in (2.7). Let $I \subseteq J$ be ideals such that J/I is cyclic and $J, (I : J) \in \mathcal{F}$. Then J and $J \cdot (I : J)$ are idempotent, so the Factorization Theorem (3.12)(4) yields $I = J \cdot (I : J) \in \mathcal{F}$, as desired.

For the last statement in (3.24), we specialize to the case where $\mathcal{F}$ is the family of *all* idempotent ideals (which is certainly monoidal). Take any $M \in \mathrm{Max}(\mathcal{F}')$. We know that M is prime so R/M is an integral domain. But every ideal in R/M is idempotent, so R/M must be a field. This means that $M \in \mathrm{Max}(R)$. $\square$

By taking the intersection of the family of idempotent ideals with the family of f.g. ideals, we get the family of ideals in R that are generated by single idempotents. These are, of course, just the ideal direct summands of R. Thus, (3.17) (with S chosen to be the set of all idempotents) would have implied that the family of such summands is strongly Oka. Independently of this, the following direct analysis gives a still better conclusion.

Proposition 3.25. *The family $\mathcal{F}$ of ideal direct summands of R has the property* (P_3). *Any $M \lhd R$ that is maximal with respect to not being a direct summand of R is a maximal ideal. A ring R is semisimple iff all maximal ideals in R are direct summands of R.*

Proof. Let $A, B \in \mathcal{F}$; say $A = eR$ and $B = fR$, where e, f are idempotents. Then $A \cap B = AB = efR \in \mathcal{F}$. This clearly implies that $\mathcal{F}$ satisfies (Q_3) and therefore (P_3), according to (2.7). Now let $M \in \mathrm{Max}(\mathcal{F}')$. By (2.4), M is prime. In the meantime, every ideal in R/M is a direct summand, so R/M is a semisimple ring. Since R/M is an integral domain, it must be a field, and so $M \in \mathrm{Max}(R)$.

The last statement in the proposition now follows as in the proof of (3.22) (this time by applying Zorn's Lemma to $\mathcal{F}'$).[3] $\square$

Remark 3.26. In general, the family $\mathcal{F}$ above does not have the property (P_2)! To see this, let $R = C \times D$, where C, D are two rings, and assume D has a nonzero ideal J with $J^2 = 0$. Then C is a direct summand of R, and for $I = C \oplus J \lhd R$, we have $I^2 = C^2 \oplus J^2 = C \subseteq I$. However, I is not a direct summand of R, for otherwise J would be a direct summand of D, which is not the case as $J^2 = 0 \neq J$.

[3] The last statement in (3.25), with "maximal ideal" replaced by "maximal right ideal," is a well-known characterization of (possibly noncommutative) semisimple rings: it can be easily checked directly by an argument using the right socle of the module R_R.

4. Categories of cyclic modules

For a commutative ring R, we will write $\mathfrak{M}(R)$ for the category of (say left) R-modules, and $\mathfrak{M}_c(R)$ for the full subcategory of *cyclic* R-modules. There is a natural one-one correspondence between the ideals of R and the isomorphism classes of the objects of $\mathfrak{M}_c(R)$. To an ideal $I \lhd R$, we associate the isomorphism class of the cyclic R-module R/I. To a cyclic module M in $\mathfrak{M}_c(R)$, we associate the ideal $\mathrm{ann}(M) \lhd R$, which depends only on the isomorphism class of M. Using this one-one correspondence, we can essentially identify the family of ideals of R with the family of isomorphism classes of the objects of the category $\mathfrak{M}_c(R)$.

In this section, we show that the viewpoint above leads to nice interpretations of Oka and Ako ideal families in R (and their strong analogues) in terms of certain subcategories of $\mathfrak{M}_c(R)$. To handle first the Oka case, we will say that a subcategory[4] $\mathcal{C}$ of $\mathfrak{M}_c(R)$ is *closed under extensions* if it contains the zero module and, for any exact sequence $0 \to L \to M \to N \to 0$ in $\mathfrak{M}_c(R)$, $L, N \in \mathcal{C}$ implies $M \in \mathcal{C}$. From this condition, it follows that, whenever $L' \cong L$, $L \in \mathcal{C}$ implies $L' \in \mathcal{C}$ (since there exists an exact sequence $0 \to L \to L' \to 0 \to 0$).

Theorem 4.1. *Let $\mathcal{C}$ be a subcategory of $\mathfrak{M}_c(R)$ that is closed under extensions. Then*

$$\mathcal{F}_{\mathcal{C}} := \{I \lhd R\colon\ R/I \in \mathcal{C}\} \tag{4.2}$$

is an Oka family. Conversely, let $\mathcal{F}$ be an Oka family of ideals in R. Then

$$\mathcal{C}_{\mathcal{F}} := \big\{M \in \mathfrak{M}_c(R)\colon\ M \cong R/I \text{ for some } I \in \mathcal{F}\big\} \tag{4.3}$$

is a subcategory of $\mathfrak{M}_c(R)$ that is closed under extensions.

Proof. To begin with, $(0) \in \mathcal{C} \Rightarrow R \in \mathcal{F}_{\mathcal{C}}$. To see that $\mathcal{F}_{\mathcal{C}}$ is Oka, we check that it has the property (O_5) in (2.7). Let $I, J \lhd R$ be such that $J = (I, a)$ for some $a \in R$, with J, $(I : J) \in \mathcal{F}_{\mathcal{C}}$; that is, R/J, $R/(I : J) \in \mathcal{C}$. Consider the exact sequence

$$0 \to J/I \to R/I \to R/J \to 0 \tag{4.4}$$

in M. Since $(I : J) = (I : a)$, we have an R-isomorphism $R/(I : J) \cong J/I$ defined by $\bar{1} \mapsto \bar{a}$. Thus, $J/I \in \mathcal{C}$, and so (4.4) implies that $R/I \in \mathcal{C}$; that is, $I \in \mathcal{F}_{\mathcal{C}}$, as desired.

Conversely, let $\mathcal{F}$ be an Oka family. The fact that $R \in \mathcal{F}$ gives $(0) \in \mathcal{C}_{\mathcal{F}}$. Consider any short exact sequence

$$0 \to L \to M \to N \to 0 \quad \text{in } \mathfrak{M}_c(R), \tag{4.5}$$

where $L, N \in \mathcal{C}_{\mathcal{F}}$. We would like to show that $M \in \mathcal{C}_{\mathcal{F}}$. To this end, represent M (up to an isomorphism) in the form R/I, for some $I \lhd R$. We may take L to be in the form J/I where $J = (I, a)$ for some $a \in R$. Then $R/J \cong N \in \mathcal{C}_{\mathcal{F}} \Rightarrow J \in \mathcal{F}$. As before, we also have $R/(I : J) \cong J/I = L \in \mathcal{C}_{\mathcal{F}}$, so $(I : J) \in \mathcal{F}$. The fact that $\mathcal{F}$ is Oka now gives $I \in \mathcal{F}$, and hence $M \cong R/I \in \mathcal{C}_{\mathcal{F}}$, as desired. □

[4] The word "subcategory" shall always mean "full subcategory" in the rest of this paper.

As it turned out, it is also possible to give an interpretation of the Ako property of an ideal family $\mathcal{F}$ by working with the subcategory $\mathcal{C}_{\mathcal{F}}$ associated with it. The result here is as follows.

Theorem 4.6. *A subcategory $\mathcal{C}$ of $\mathfrak{M}_c(R)$ containing the zero module is said to be Ako if, for any two short exact sequences $0 \to L_i \to M \to N_i \to 0$ $(i = 1, 2)$ in $\mathfrak{M}_c(R)$ such that there exist surjections $N_2 \twoheadrightarrow L_1$ and $N_1 \twoheadrightarrow L_2$, we have $N_1, N_2 \in \mathcal{C} \Rightarrow M \in \mathcal{C}$. If $\mathcal{C}$ is Ako, then $\mathcal{F}_{\mathcal{C}}$ is an Ako family. Conversely, if an ideal family $\mathcal{F}$ in R is Ako, then so is the associated subcategory $\mathcal{C}_{\mathcal{F}} \subseteq \mathfrak{M}_c(R)$.*

Proof. First assume $\mathcal{C}$ is Ako. To show that $\mathcal{F}_{\mathcal{C}}$ is Ako, we check the property (Q_5) in (2.7). For this, we start with $AB \subseteq I \subseteq A \cap B$, where $A, B \in \mathcal{F}_{\mathcal{C}}$, $I \lhd R$, and $A/I, B/I \in \mathfrak{M}_c(R)$. Consider the two exact sequences

$$0 \to A/I \to R/I \to R/A \to 0 \quad \text{and} \quad 0 \to B/I \to R/I \to R/B \to 0. \tag{4.7}$$

Since A/I is generated by some $\bar{a}$, we have a surjection $R/B \twoheadrightarrow A/I$ defined by $\bar{1} \mapsto \bar{a}$ (noting that $AB \subseteq I$). Similarly, we have a surjection $R/A \twoheadrightarrow B/I$. As $R/A, R/B \in \mathcal{C}$, the Ako property on $\mathcal{C}$ implies that $R/I \in \mathcal{C}$. This translates into $I \in \mathcal{F}_{\mathcal{C}}$, which checks ($Q_5$) for $\mathcal{F}_{\mathcal{C}}$.

Conversely, let $\mathcal{F}$ be an Ako ideal family. To check that $\mathcal{C} := \mathcal{C}_{\mathcal{F}} \subseteq \mathfrak{M}_c(R)$ has the Ako property, consider the two exact sequences (in $\mathfrak{M}_c(R)$) in the statement of (4.6), with given surjections $N_2 \twoheadrightarrow L_1$ and $N_1 \twoheadrightarrow L_2$, and with $N_1, N_2 \in \mathcal{C}$. Since $M \in \mathfrak{M}_c(R)$, $M \cong R/I$ for some $I \lhd R$. There exist ideals $A, B \supseteq I$ such that we can "identify" the given sequences with those in (4.7). Then, $N_1, N_2 \in \mathcal{C}$ amount to $A, B \in \mathcal{F}$. A surjection $N_2 \twoheadrightarrow L_1$ means now a surjection $R/B \twoheadrightarrow A/I$. This implies that A/I is cyclic and $AB \subseteq I$. Similarly, B/I is also cyclic. Then (Q_5) implies that $I \in \mathcal{F}$, and thus $M \cong R/I \in \mathcal{C}_{\mathcal{F}} = \mathcal{C}$. □

By slightly tweaking the hypotheses of (4.6) but using the same arguments, we can formulate a similar categorical characterization for the strong Ako property for ideal families. However, this time, *we can no longer stay completely within the category $\mathfrak{M}_c(R)$,* and must work with the full module category $\mathfrak{M}(R)$. This is due to the fact that, in the condition (Q_4) in (2.7) characterizing the strong Ako property, the module B/I is *not* assumed to be cyclic. Nevertheless, we note that, in the second part of the proof of (4.6), the existence of a *single* surjection $R/B \twoheadrightarrow A/I$ is sufficient to imply that A/I is cyclic *and* $AB \subseteq I$. Thus, reworking the proof of (4.6) leads to the following characterization of strongly Ako ideal families in R, in parallel to (4.6).

Theorem 4.8. *The strong Ako property for an ideal family $\mathcal{F}$ corresponds to the following "strong Ako property" on its associated category $\mathcal{C} \subseteq \mathfrak{M}_c(R)$: $(0) \in \mathcal{C}$, and for any two exact sequences in* (4.6) *where all modules except L_2 are cyclic, if there exists a surjection $N_2 \twoheadrightarrow L_1$, then $N_1, N_2 \in \mathcal{C} \Rightarrow M \in \mathcal{C}$.*

One interesting consequence of (4.8) is the following.

Corollary 4.9. *Let $\mathcal{F}$ be a semifilter of ideals in R. Then $\mathcal{F}$ is Oka iff it is strongly Ako.*

Proof. From (2.7), we know that the "if" part is true even without $\mathcal{F}$ being a semifilter. Now assume $\mathcal{F}$ is an Oka semifilter. *The semifilter property means that the associated category $\mathcal{C} := \mathcal{C}_{\mathcal{F}}$ is closed with respect to quotients.* To check that $\mathcal{C}$ has the strong Ako property, consider two exact sequences as in the statement of (4.8), where all modules except L_2 are cyclic, and there exists a surjection $N_2 \twoheadrightarrow L_1$. Assume that $N_1, N_2 \in \mathcal{C}$; then $L_1 \in \mathcal{C}$ too. From the first exact sequence, and the fact (from (4.1)) that $\mathcal{C}$ is closed under extensions in $\mathfrak{M}_c(R)$, we see that $M \in \mathcal{C}$, as desired. $\square$

Of course, it is also possible to prove (4.9) directly from the definitions (of "Oka" and "strongly Ako"); see [LR]. However, the categorical characterizations of these properties made (4.9) a very natural result. (A similar remark applies to Corollary 4.12 below.)

The case of ideal families with the property (Q_3) (or equivalently, (P_3)) can be given a categorical characterization as well. In this characterization, however, there will be no surjections $N_2 \twoheadrightarrow L_1$ or $N_1 \twoheadrightarrow L_2$, so we have to treat the condition $AB \subseteq I$ in (Q_3) with a little more care. Nevertheless, the same argument for proving (4.6) can be easily adapted to yield the following characterization result, where $\operatorname{ann}(N)$ denotes the R-annihilator of an R-module N.

Theorem 4.10. *The* (Q_3) *property for an ideal family $\mathcal{F}$ corresponds to the following* "(Q_3) *property*" *on its associated category $\mathcal{C} \subseteq \mathfrak{M}_c(R)$: $(0) \in \mathcal{C}$, and for any two surjections $M \twoheadrightarrow N_i$ $(i = 1, 2)$ in $\mathfrak{M}_c(R)$, if $\operatorname{ann}(N_1) \cdot \operatorname{ann}(N_2) \subseteq \operatorname{ann}(M)$, then $N_1, N_2 \in \mathcal{C} \Rightarrow M \in \mathcal{C}$.*

We now finish with a categorical characterization of strongly Oka ideal families.

Theorem 4.11. *An ideal family $\mathcal{F}$ is strongly Oka iff its associated category $\mathcal{C} \subseteq \mathfrak{M}_c(R)$ has the following "strongly Oka property": $(0) \in \mathcal{C}$, and, for any R-module exact sequence $0 \to L \to M \to N \to 0$ where $M \in \mathfrak{M}_c(R)$, if $N \in \mathcal{C}$ and $\operatorname{ann}(L) \in \mathcal{F}_{\mathcal{C}}$, then $M \in \mathcal{C}$.*

Proof. No new ideas are needed for this proof, if we just keep in mind that, for any two ideals $I \subseteq J$ in R, $\operatorname{ann}(J/I) = (I : J)$. $\square$

Corollary 4.12. *Let $\mathcal{F}$ be a semifilter of ideals in R. Then $\mathcal{F}$ is strongly Oka iff it has the* (Q_3) *property.*

Proof. Again by (2.7), the "if" part is true without $\mathcal{F}$ being a semifilter. Now assume $\mathcal{F}$ is a strongly Oka semifilter. To check that $\mathcal{C} := \mathcal{C}_{\mathcal{F}}$ has the (Q_3) property in the sense of (4.10), consider a cyclic module (say) $M = R/I$ ($I \lhd R$), and two epimorphic images, say R/A and R/B, with $A, B \supseteq I$ and $AB \subseteq I$. Assume that $R/A, R/B \in \mathcal{C}$, so that $A, B \in \mathcal{F}$. Then $L := \ker(R/I \twoheadrightarrow R/B) = B/I$. Since $\operatorname{ann}(L) = \operatorname{ann}(B/I) \supseteq A$, we have $\operatorname{ann}(L) \in \mathcal{F}$, so (4.11) implies that $M \in \mathcal{C}$, as desired. $\square$

After going through all the results in this section, we can see with hindsight why the Oka and Ako properties are particularly nice to single out in the study of the Prime Ideal Principle: they possess the simplest categorical characterizations *purely within the category $\mathfrak{M}_c(R)$ of cyclic R-modules.* As for the Oka and Ako properties themselves, a direct comparison of (4.1) and (4.6) bears out once more our remark (made after (3.18)) that "Oka" is the superior one, since its categorical characterization in (4.1) is, in turn, simpler and nicer than that for "Ako" in (4.6). Indeed, our work in the next section will make it quite clear that the easy categorical characterization of "Oka" in (4.1) can be used very effectively to construct (by going over to the full module category $\mathfrak{M}(R)$) a good number of interesting new examples of Oka families over arbitrary rings.

5. Applications of the categorical viewpoint

According to Theorem 4.1, the choice of an Oka family in a ring R is equivalent to the choice of a subcategory of $\mathcal{C} \subseteq \mathfrak{M}_c(R)$ that is closed under extensions in $\mathfrak{M}_c(R)$. From a constructive viewpoint, we can thus start with any such subcategory $\mathcal{C}$, and use it to produce an Oka ideal family in R. A natural way to find good candidates for $\mathcal{C}$ is the following. Let $\mathcal{E}$ be any subcategory of $\mathfrak{M}(R)$ that is closed under extensions (in $\mathfrak{M}(R)$). If we define $\mathcal{C} := \mathcal{E} \cap \mathfrak{M}_c(R)$, it is easy to check that $\mathcal{C}$ is closed under extensions in $\mathfrak{M}_c(R)$. In this way, *any $\mathcal{E}$ above will produce an Oka ideal family in R*. In this section, we will record the many consequences of this general construction.

To begin with, we will first "account for" some of the Oka families constructed in Section 3 from the categorical viewpoint. The first example is based on a given (nonempty) multiplicative set $S \subseteq R$. For any R-module M, let M_t be the *S-torsion submodule* of M, as defined in (3.7). We will say that M is *S-torsion* if $M_t = M$. A routine check shows that the subcategory $\mathcal{E} \subseteq \mathfrak{M}(R)$ consisting of all S-torsion modules (and morphisms between them) is closed under extensions (in $\mathfrak{M}(R)$). Specializing to cyclic modules, it is also easy to see that, for $I \lhd R$, $R/I \in \mathcal{E} \Leftrightarrow I \cap S \neq \emptyset$. Therefore, in the language of (4.1), the category $\mathcal{C} := \mathcal{E} \cap \mathfrak{M}_c(R)$ corresponds to the ideal family $\{I \lhd R\colon\ I \cap S \neq \emptyset\}$, which is exactly the (monoidal) Oka family constructed in (3.1).

Somewhat surprisingly, in the context above, a *second* Oka family can be constructed, and this will actually lead to some interesting new information. We define an R-module M to be *S-torsionfree* if $M_t = 0$. Again, the category $\mathcal{E}_0$ of such modules is easily seen to be closed under extensions in $\mathfrak{M}(R)$. Therefore, $\mathcal{C}_0 = \mathcal{E}_0 \cap \mathfrak{M}_c(R)$ is closed under extensions in $\mathfrak{M}_c(R)$. Now, for $I \lhd R$, it is routine to check that $R/I \in \mathcal{E}_0$ iff I is *S-saturated* in the sense that $I = \mathrm{Sat}_S(I)$, where

$$\mathrm{Sat}_S(I) = \{r \in R\colon\ sr \in I \text{ for some } s \in S\}.^5 \tag{5.1}$$

Indeed, it is true in general that $(R/I)_t = \mathrm{Sat}_S(I)/I$. We can now deduce easily the following result.

Proposition 5.2. *The family $\mathcal{F}_0$ of S-saturated ideals in R is an Oka family that is closed with respect to* (*arbitrary*) *intersections. The maximal members of $\mathcal{F}_0'$ are maximal ideals in R*; *in fact,*

$$\mathrm{Max}\big(\mathcal{F}_0'\big) = \big\{\mathfrak{m} \in \mathrm{Max}(R)\colon\ \mathfrak{m} \cap S \neq \emptyset\big\}. \tag{5.3}$$

This set is nonempty iff S contains a nonunit.

Proof. To begin with, it is clear that $I, J \in \mathcal{F}_0 \Rightarrow I \cap J \in \mathcal{F}_0$. Since $\mathcal{F}_0$ corresponds to $\mathcal{C}_0$, the fact that $\mathcal{C}_0$ is closed under extensions in $\mathfrak{M}_c(R)$ guarantees that $\mathcal{F}_0$ is Oka. Let $I \in \mathrm{Max}(\mathcal{F}_0')$. The Prime Ideal Principle (2.4) guarantees that I is prime. But $I \neq \mathrm{Sat}_S(I) \Rightarrow sr \in I$ for some $r \notin I$ and $s \in S$. Since I is prime, we have $s \in I \cap S$. Now the only S-saturated ideal containing s is the full ring R. Thus, the only ideal properly containing I is R, which shows that $I \in \mathrm{Max}(R)$.

[5] Alternatively, $\mathrm{Sat}_S(I)$ can be defined to be the contraction of the extension of I with respect to the localization map $R \to S^{-1}R$.

This proves the inclusion "$\subseteq$" in (5.3). Conversely, if $\mathfrak{m} \in \mathrm{Max}(R)$ contains an element of S, obviously $\mathrm{Sat}_S(\mathfrak{m}) = R \neq \mathfrak{m}$. Thus, $\mathfrak{m} \in \mathcal{F}_0'$, which of course means that $\mathfrak{m} \in \mathrm{Max}(\mathcal{F}_0')$. The last conclusion is clear from Eq. (5.3). □

Next, let us quickly account for the Oka families of principal ideals, f.g. ideals, and essential ideals in reduced rings from the categorical viewpoint. In the following, $\mathcal{F}$ shall refer to an Oka family in R, and $\mathcal{C}$ shall refer to its corresponding subcategory in $\mathfrak{M}_c(R)$.

Proposition 5.4.

(1) *If $\mathcal{F}$ is the Oka family of all f.g. ideals in R, $\mathcal{C}$ is the category of all finitely presented cyclic R-modules.*
(2) *If $\mathcal{F}$ is the Oka family of all principal ideals in R, $\mathcal{C}$ is the category of all* 1*-generator* 1*-relator R-modules.*
(3) *If $\mathcal{F}$ is the Oka family of all principal ideals generated by non* 0*-divisors in R, $\mathcal{C}$ is the category of all nonfree cyclic R-modules with a free resolution*[6] *of length* 1.
(4) *Let R be a reduced ring. If $\mathcal{F}$ is the Oka family of all essential ideals in R, $\mathcal{C}$ is the category of all cyclic singular R-modules.*[7]

Proof. (1) If $I \lhd R$ is f.g., $0 \to I \to R \to R/I \to 0$ shows that R/I is finitely presented. Conversely, if R/I is finitely presented, [La$_2$, (4.26)(b)] implies that I is f.g. (Recall that the category $\mathcal{E}$ of all finitely presented R-modules is closed under extensions in $\mathfrak{M}(R)$ [La$_4$, Exercise 4.8(2)]. In view of (4.1), this gives a new *categorical* proof for Oka's *Corollaire* 2 that $\mathcal{F}$ is an Oka family.)

(2) is clear. For (3), if $I = (a)$ where a is a non 0-divisor, R/I is nonfree and $0 \to (a) \to R \to R/I \to 0$ is a free resolution of length 1. Conversely, if R/I is nonfree and has a free resolution $0 \to R^m \to R^n \to R/I \to 0$, Schanuel's Lemma gives $I \oplus R^n \cong R^m \oplus R$. Since $I \neq 0$, we must have $m = n$ and $I \cong R$ [La$_4$, Exercise 5.16], so $I = (a)$ for some non 0-divisor $a \in R$.

(4) This follows from the fact that $I \lhd R$ *is essential iff R/I is a singular R-module.* (This fact holds over any ring R: see [La$_4$, Exercise 7.2(b)]. However, we need an assumption such as R is reduced to guarantee that the category $\mathcal{E}$ of singular R-modules is closed under extensions in $\mathfrak{M}(R)$; see [Go, (1.23)].) □

Another very effective way of constructing subcategories $\mathcal{E} \subseteq \mathfrak{M}(R)$ that are closed under extensions is the following. Suppose R is a k-algebra, where k is a fixed commutative ring. Let (E) be a k-module property that is satisfied by the zero module and preserved by k-module extensions. Given (E), we can define $\mathcal{E}$ to be the category of R-modules that have the property (E) *when viewed as k-modules.* Clearly, $\mathcal{E}$ is closed under extensions in $\mathfrak{M}(R)$, since exact sequences in $\mathfrak{M}(R)$ remain exact in $\mathfrak{M}(k)$. Thus, the general construction $\mathcal{C} = \mathcal{E} \cap \mathfrak{M}_c(R)$ introduced at the beginning of this section is applicable. Several immediate choices of the property (E) come to mind:

(5.5) *k-injectivity*: the property of being injective as a k-module.
(5.6) *k-projectivity*: the property of being projective as a k-module.

[6] All free resolutions are assumed to use only free modules of finite rank.
[7] An R-module M is said to be *singular* if every $m \in M$ has an essential annihilator in R.

(5.7) *k-flatness*: the property of being flat as a k-module.
(5.8) *the property of having a finite free resolution* (FFR) *in* $\mathfrak{M}(k)$.

Indeed, let $0 \to L \to M \to N \to 0$ be exact in $\mathfrak{M}(k)$. If L, N are k-injective (respectively k-projective), then the sequence splits, so $M \cong L \oplus N$ is also k-injective (respectively k-projective). If L, N are k-flat instead, [La_2, (4.13)] implies that M is k-flat. If L, N have FFRs, [La_4, (5.21)(C)] shows that M has FFR. Thus, the choices of (E) in (5.5)–(5.8) all lead to Oka families $\mathcal{F}$ in R; in particular, the conclusions of (2.4) and (2.6) both apply to $\mathcal{F}$. Note that, in the case where $k = R$, the $\mathcal{F}$ arising from (5.6) is precisely the family of direct summands of R (discussed in (3.25)); on the other hand, the $\mathcal{F}$ arising from (5.7) is *the family of all pure ideals in R,* according to [La_2, (4.86)].[8]

For further generalization of (5.5)–(5.6), we can take (E) to be the property of having k-projective (respectively k-injective) dimension $< n$, where n is a fixed positive integer, or the symbol ∞.[9] For $n = 1$, we get back (5.5)–(5.6), but for $n = 2$ and R a k-projective k-algebra, we get the new Oka family of k-projective ideals in the ring R. In the latter case, we see, for instance, that *an ideal in R maximal with respect to not being k-projective is prime.* And in the case $k = R$, a direct application of (2.6)(3) yields the following characterization of noetherian hereditary rings.

Corollary 5.9. *A noetherian ring is hereditary iff its prime ideals are all projective.*

This characterization can actually be further sharpened by replacing the word “prime” by “maximal”. However, the proof of this would require more sophisticated tools from the homological dimension theory of noetherian rings, such as [La_2, (5.92)].

Turning to finiteness conditions, we can also choose (E) to be any one of the following k-module properties:

(5.10) *the property of being a finitely cogenerated k-module*;[10]
(5.11) *the property of being a finite* (*respectively f.g.*) *k-module*;
(5.12) *the property of being a noetherian* (*respectively artinian*) *k-module*;
(5.13) *the property of being a k-module of finite length* (*or such a module whose composition factors have certain prescribed isomorphism types, etc.*).

Using these properties for (E), we get new examples of Oka families $\mathcal{F}$ in R. (The families $\mathcal{F}$ resulting from (5.11)–(5.13) are clearly semifilters, so (4.9) implies that they are even *strongly Ako* families!) Each of these families $\mathcal{F}$ leads to a new application of the Prime Ideal Principle (2.4). For instance, from (5.11) and (5.12), we get the following results without any further proof:

(5.14) *An ideal $I \lhd R$ maximal with respect to to having infinite index in R is prime.*

---

[8] An ideal $I \lhd R$ is said to be *pure* if $I \hookrightarrow R$ remains an inclusion upon tensoring with any R-module. It follows from the analysis here that *all pure ideals in R form an Oka family, and that an ideal maximal with respect to not being pure is prime.*

[9] For the fact that the property (E) is preserved by k-module extensions, see, for instance, [La_4, Exercise (5.0)(b)].

[10] For the definition of finitely cogenerated k-modules, see [La_2, (19.2)]. The fact that the category of such k-modules is closed under extensions in $\mathfrak{M}(k)$ is proved in [La_4, Exercise (19.4)].

(5.15) *An ideal $I \lhd R$ maximal with respect to R/I not being an artinian k-module is maximal* (noting that an artinian domain is a field).

(5.16) *An ideal $I \lhd R$ maximal with respect to R/I not being a noetherian k-module is prime.* This is Exercise 14 in [Ka_2, p. 54], which required considerable work, but can be used to prove the Eakin–Nagata Theorem (for the descent of the noetherian ring property) in [Ea,Na_2].

In the case $k = R$, if we choose (E) to be the property (5.10), the Prime Ideal Principle Supplement (2.6)(3) can be used to deduce the following nice "artinian version" of I.S. Cohen's Theorem.

Corollary 5.17. *A ring R is artinian iff, for every prime ideal $P \lhd R$, P is finitely generated and R/P is finitely cogenerated.*

Proof. This follows from (2.6)(3), Cohen's (noetherian) theorem, and the well-known fact that a module M is artinian iff all factor modules of M are finitely cogenerated. (For a proof of the latter, see [La_2, Exercise (19.0)].) $\square$

Note that, in this corollary, the condition "P is f.g." cannot be removed (for the "if" part). For instance, if $(R, \mathfrak{m})$ is a 0-dimensional local ring, then R/P is finitely cogenerated for all prime ideals P, but R need not be artinian. (Of course, R is artinian iff $\mathfrak{m}$ is f.g.)

In conclusion, let us mention one more class of examples. Suppose the k-module property (E) studied in this section is preserved not only by extensions but also by quotients (e.g. (5.11)–(5.13)). Then the following construction is possible. Fix *any* R-module M and define $\mathcal{F}$ to be the family of ideals $I \lhd R$ such that, for every R-submodule $L \subseteq M$, L/IL has the property (E) when viewed as a k-module. The two properties assumed on (E) easily imply that *$\mathcal{F}$ is a monoidal filter.* Thus, (2.4) and (2.6) apply again to $\mathcal{F}$. This example is inspired by a construction of Isaacs in [Ka_2, p. 74], where the property (E) is taken to be (5.12) (with $k = R$).

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