Math 54 Handout 8

July 2, 2018

Question 1.

$$\begin{pmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 6 & -8 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -6x_3 + 8x_4 - x_5 \\ 2x_3 - x_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

So

$$Null(A) = Span \left\{ \begin{pmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Question 2.

$$\begin{pmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 2 & 1 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 0 & -5 & -20 & 15 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}$$

is a basis for the subspace spanned by the four vectors. Thus the dimension is 2.

Question 3.

Writing the four Laguerre polynomials in coordinates and putting them as columns of a matrix, we have

$$\begin{pmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 19 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Since this matrix is invertible (determinant 1), we see that the four column vectors are linearly independent and span $\mathbb{R}^4$, so $1, 1-t, 2-4t+t^2, 6-18t+19t^2-t^3$ form a basis of $P_3[t]$.

To compute the coordinates, augment and solve:

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 6 & 7 \\ 0 & -1 & -4 & -18 & -8 \\ 0 & 0 & 1 & 19 & 3 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

so the coordinates is just $\begin{pmatrix} 5 \\ -4 \\ 3 \\ 0 \end{pmatrix}$

Question 4.

True. First of all, $0 \in V$ and $0 \in W$, so $0 \in V \cap W$. If $v \in V \cap W$ and $w \in V \cap W$, then $v + w$ is in V and also in W, so $v + w \in V \cap W$. Similarly, if $v \in V \cap W$, then cv is in V and also in W, so $cv \in V \cap W$.

Question 5.

1. $$T(A+B) = (A+B) + (A+B)^T = (A+A^T) + (B+B^T) = T(A) + T(B)$$

   and

   $$T(cA) = (cA) + (cA)^T = c(A+A^T) = cT(A)$$

2. $$T(\frac{1}{2}B) = \frac{1}{2}B + \frac{1}{2}B^T = \frac{1}{2}B + \frac{1}{2}B = B$$

3. In part 2 we have shown that if $B = B^T$ then it is in the image of T. So it remains to show that the image of T satisfies $B = B^T$. The image is of the form $(A + A^T)$, and $(A + A^T)^T = A^T + (A^T)^T = (A + A^T)$, and so we conclude that the image of T is precisely all matrices B such that $B = B^T$.

4. The kernel of T consists of all matrices A such $A = -A^T$. Note that these matrices must have only 0 on the diagonal.