Math 54 Handout 3 Solution

June 20, 2018

Question 1.

$$A = \begin{pmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -4 & -2 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -4 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

here x_2, x_4, and x_6 are free variables, so the solution looks like

$$\begin{cases} x_1 & = 4x_2 - 5x_6 \\ x_3 & = x_6 \\ x_5 & = 4x_6 \end{cases}$$

Writing the solution in vector form gives

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 4x_2 - 5x_6 \\ x_2 \\ x_6 \\ x_4 \\ 4x_6 \\ x_6 \end{pmatrix} = x_2 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{pmatrix}$$

Question 2.

$$\left(\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_3 is free here, so the solution looks like

$$\begin{cases} x_1 = -4x_3 - 5 \\ x_2 = 3x_3 + 3 \end{cases}$$

Writing it in vector form, the solution looks like

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} -4x_3 - 5 \\ 3x_3 + 3 \\ x_3 \end{pmatrix} \to \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$$

Question 3.

The first question amounts to checking if there is a column without pivots, while the second part asks to check if there is a pivot in every row. So

1. A is a 4×3 matrix with three pivots. No, No.
2. A is a 3×4 matrix with three pivots. Yes, Yes.
3. A is a 3×3 matrix with two pivots. Yes, No.
4. A is a 2×2 matrix with two pivots. No, Yes.

Question 4.

$$\begin{pmatrix} 1 & 3 & -1 \\ -1 & -5 & 5 \\ 4 & 7 & h \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 \\ 0 & -2 & 4 \\ 0 & -5 & h+4 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 \\ 0 & -1 & 2 \\ 0 & -5 & h+4 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & h-6 \end{pmatrix}$$

Since we want them to be linearly independent, we want there to be pivots in every column, so we need $h - 6 \neq 0$.

Question 5.

Since the two column vectors are not multiples of each other, they are linearly independent, and so for this 4×2 matrix, there are two pivots. Thus the echelon form must be

$$\begin{pmatrix} * & * \\ 0 & * \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Question 6.

True. Because any nonzero dependence relation $a_1v_1 + a_2v_2 + a_3v_3 = 0$ where some $a_i \neq 0$ is automatically a dependence relation of $\{v_1, v_2, v_3, v_4\}$.

Question 7.

False. Counterexample: when $v_4 = 0$, $\{v_1, v_2, v_3, v_4\}$ is dependent.

Question 8.

True. Suppose there is a relation $c_1v_1 + c_2v_2 = 0$, we want to show that the dependence relation is zero, i.e. $c_1 = c_2 = 0$. We do so by multiplying on the left by the matrix A, which yiels the equation $c_1Av_1 + c_2Av_2 = A(c_1v_1 + c_2v_2) = A(0) = 0$ (this now is a dependence relation on Av_1 and Av_2). However, now as Av_1 and Av_2 are linearly independent, $c_1 = c_2 = 0$, which is what we wanted.